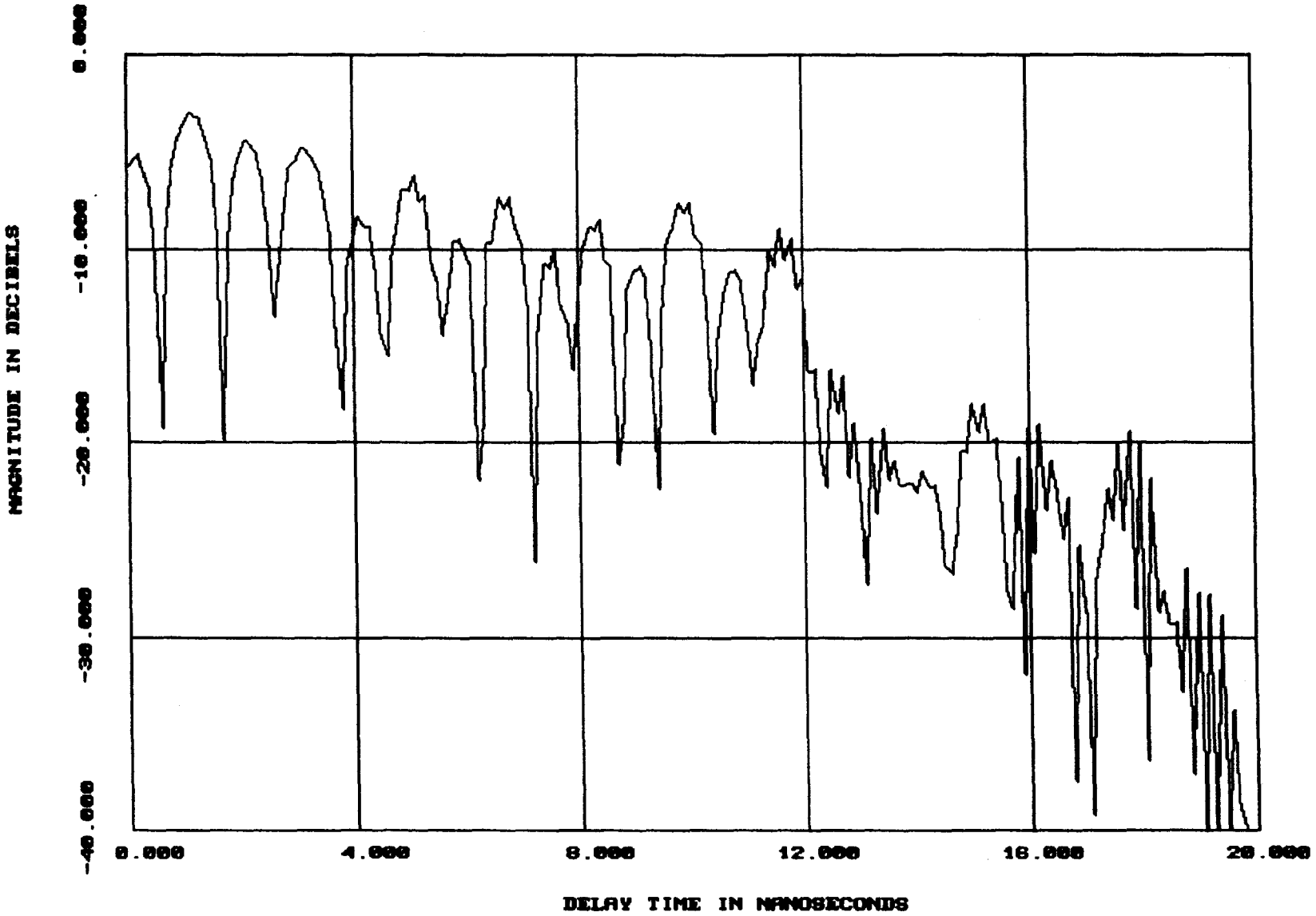


Exhibit 28. Cross correlation of Sue's Office Waveform with the Laboratory Waveform.

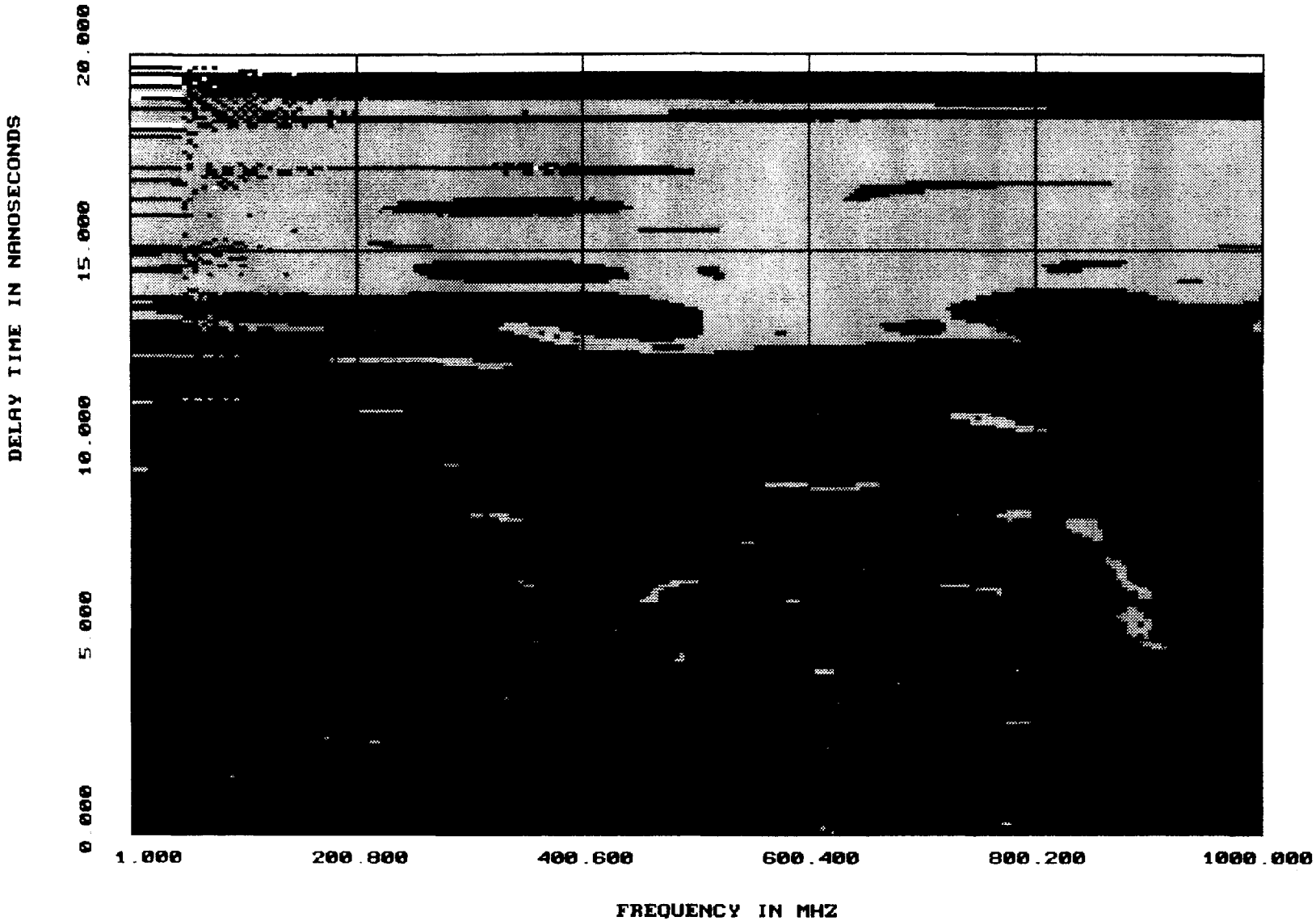


Exhibit 29. Cross ambiguity function of Sue's Office Waveform with the Laboratory Waveform. Colors show 10 dB steps.

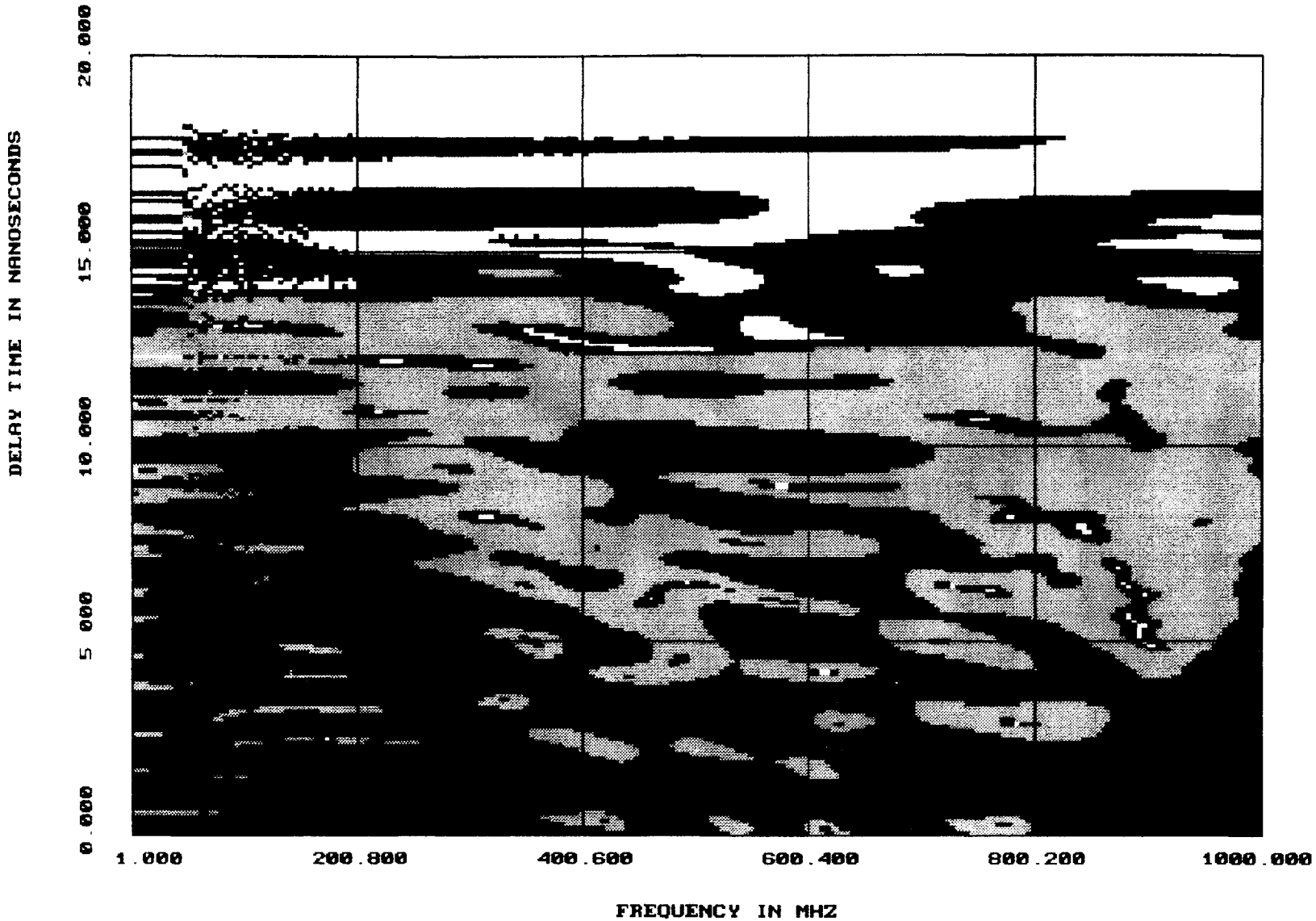


Exhibit 30. Cross ambiguity function of Sue's Office Waveform with the Laboratory Waveform. Colors show 6 dB steps.

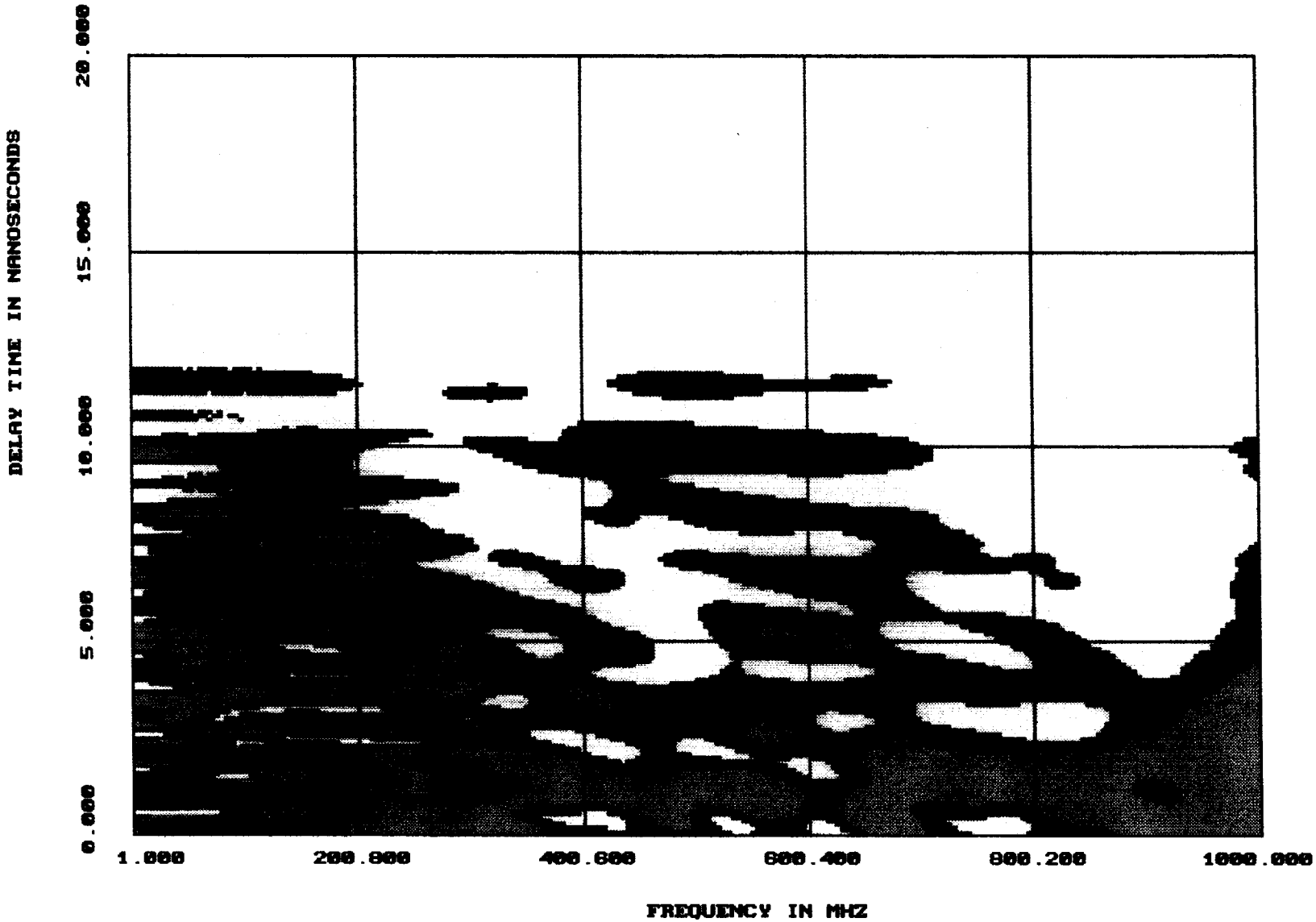
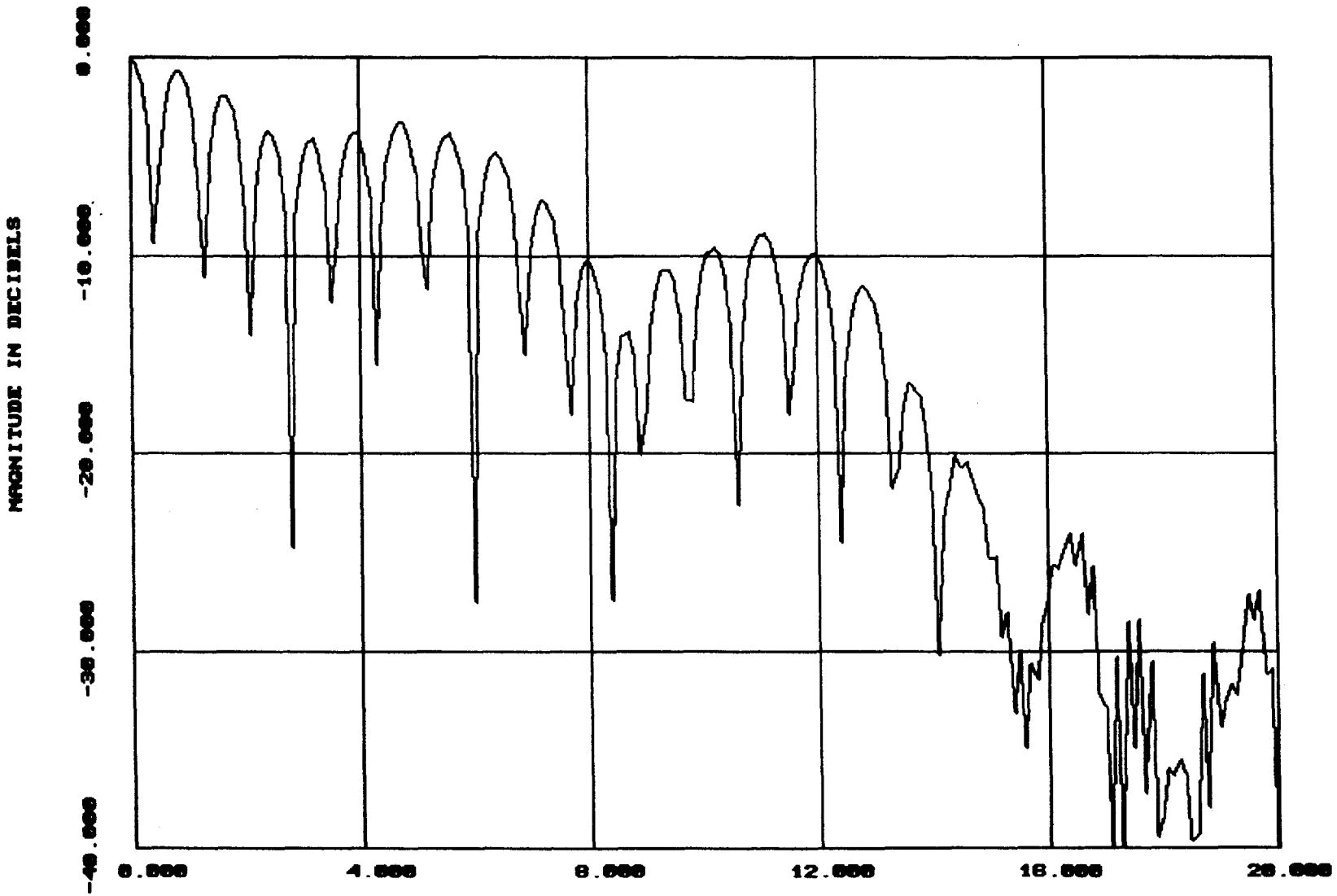


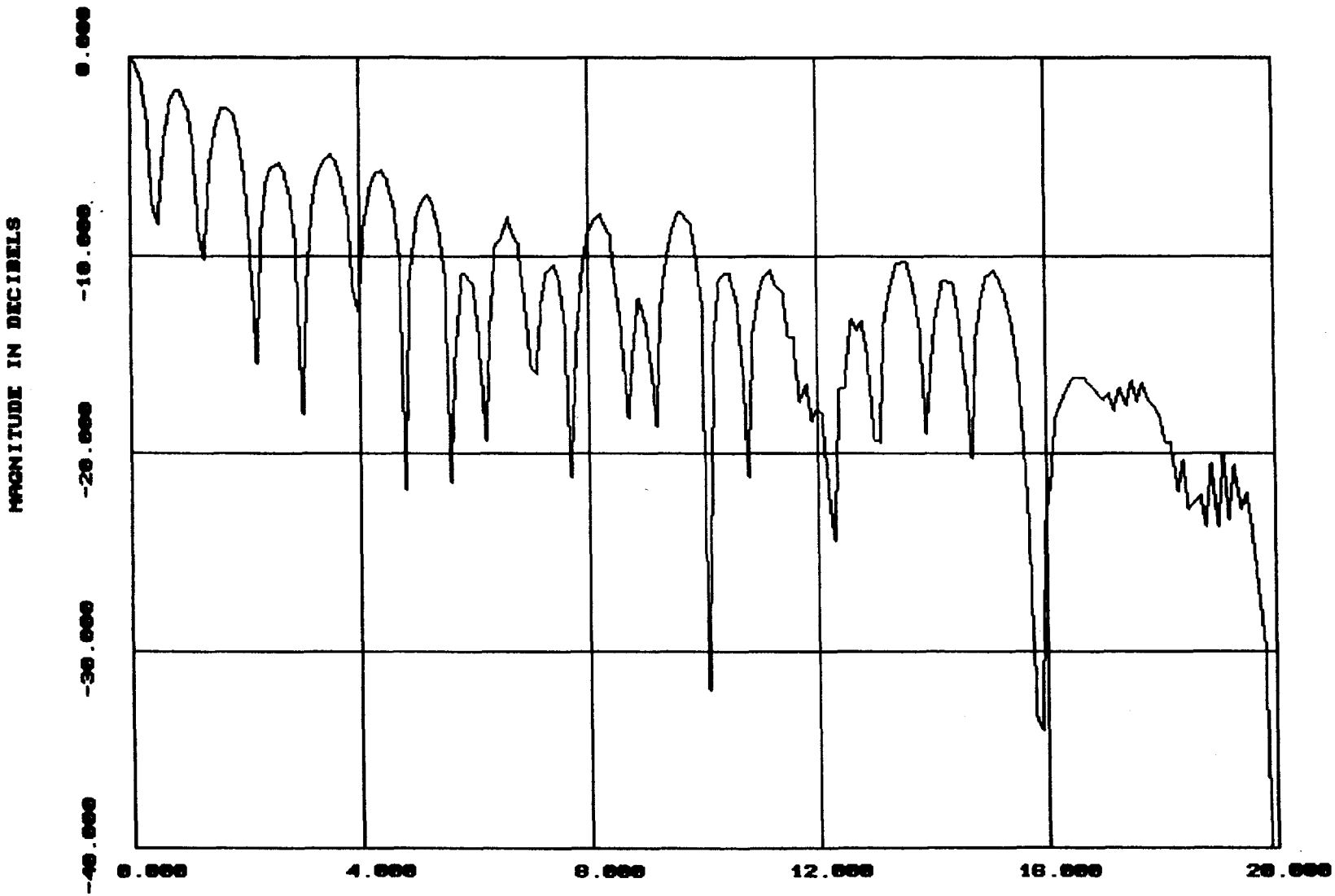
Exhibit 31. Cross ambiguity function of Sue's Office Waveform with the Laboratory Waveform. Colors show 3 dB steps.

APPENDIX 1:

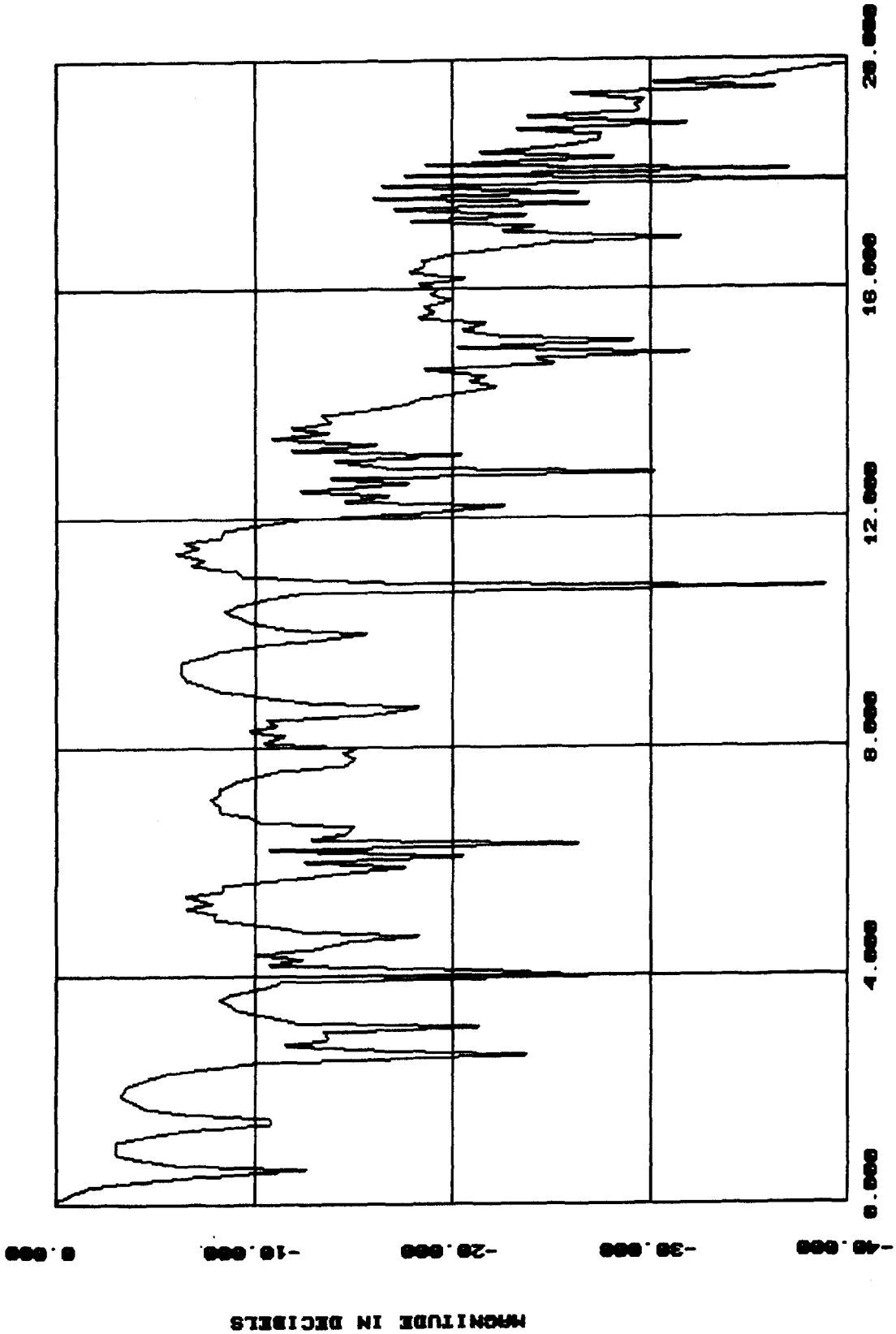
ADDITIONAL AUTOCORRELATION FUNCTIONS



NORMALIZED AUTOCORRELATION OF CHARLON.DAT



NORMALIZED AUTOCORRELATION OF MENON.DAT



NORMALIZED AUTOCORRELATION OF SUBON.DAT

APPENDIX 2:

COHERENT INTEGRATION ACCORDING TO MINKOFF

6

COHERENT AND NONCOHERENT DETECTION AND PROCESSING

The question concerning the relative advantages of coherent and noncoherent detection and processing arises in all sensing systems, and also in communication systems, and is concerned with evaluating the benefits associated with preserving phase information in the signal in the processing. In Chapter 5 it was shown that matched filtering involves a correlation of the received signal with an exact replica of itself, which therefore includes its phase. Hence matched filtering is synonymous with coherent detection, in the sense that the phase of the signal is assumed to be known exactly. There is however another form of coherent detection and processing in which the transmitted waveform consists of a train of pulses whose responses are summed, or "integrated" in a manner such that the phase relationship between successive pulses is maintained, but the absolute value of the phase may or may not be known exactly. This is referred to as coherent integration. If on the other hand the summation is performed without regard to the phase relationship between pulses the process is called non-coherent integration.

In this chapter, we first consider noncoherent detection of a single pulse and compare the results with coherent detection for which, as noted, the results have already been obtained in Chapter 5. Then the improvement in SNR yielded by coherent and non-coherent integration of a train of pulses shall be calculated and compared. Finally the performance of coherent and noncoherent integration in terms of detection and false alarm probabilities shall be analyzed and compared. A summary of the results of this chapter is presented in Section 6.5.

As is discussed in Chapter 1, the processing of sensor data may include translation of the carrier frequency down to a more manageable IF. This

step however is irrelevant to the results which follow, and to simplify the presentation is therefore omitted.

6.1 IDEAL NONCOHERENT DETECTION OF A SINGLE PULSE

Consider a situation in which the signal of interest is a rectangular carrier pulse of duration T* of the form $AP_T(t)\cos(2\pi f_0 t + \theta)$ where

$$P_T(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

The two hypotheses are

$$\begin{aligned} H_1:&\quad y(t) = s(t) + n(t) = AP_T(t)\cos(2\pi f_0 t + \theta) + n(t) \\ H_0:&\quad y(t) = n(t) \end{aligned} \tag{6.1}$$

where $n(t)$ is a realization of mean-zero, white Gaussian noise with two-sided spectral density $N_0/2$. In this case, in parallel with the discussion in Section 5.4.1, it is assumed that observations of $y(t)$ are made at the output of a hypothetical rectangular bandpass filter of bandwidth B centered at the carrier frequency f_0, as illustrated in Figure 6.1.

The bandwidth B must be sufficiently large to pass essentially all the signal energy, which will be the case if $B \ge 1/T$. In this case, by the carrier-sampling theorem, with sampling at the Nyquist rate there are BT *complex* samples in a time T and, in a manner essentially identical to that used to obtain (5.42) from (5.39) (also see Exercise 6.1), the likelihood ratio in this case is

$$\begin{aligned} &\frac{\exp[-(1/2\sigma^2)\sum_{i=1}^{BT}(y(t_i) - s(t_i))^2]}{\exp[-(1/2\sigma^2)\sum_{i=1}^{BT} y^2(t_i)]} \\ &= \frac{\exp[-(2B/2\sigma^2)\int_0^T (y(t) - A\cos(2\pi f_0 t + \theta))^2\,dt]}{\exp[-(2B/2\sigma^2)\int_0^T y^2(t)\,dt]} \end{aligned} \tag{6.2}$$

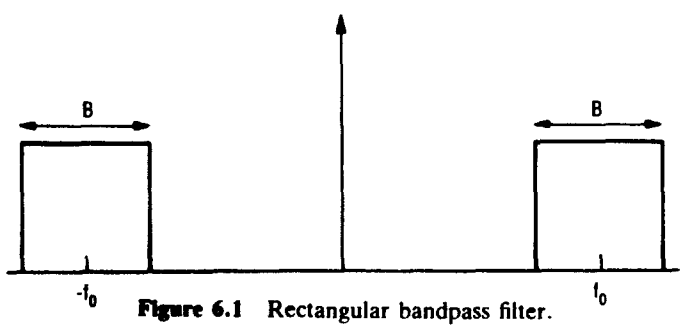


Figure 6.1 Rectangular bandpass filter.

* A generalized treatment of this subject using arbitrary complex signals is presented in Chapter 10.

As before, because the signal is of finite duration, the approximation in passing from the discrete to the continuous representation improves in the limit as B is allowed to become very large.

In the non-coherent case θ in (6.2) is unknown, and the likelihood ratio will therefore be of the form $P(y|H_1, \theta)/P(y|H_0)$, which will have a different unknown value for each value of the parameter θ. Therefore, since no auxiliary information about θ is available, it is reasonable to assume θ to be uniformly distributed over $(0, 2\pi)$ and we deal with an averaged likelihood ratio of the form

$$\begin{aligned} &\int_0^{2\pi} \frac{P(y|H_1, \theta)P(\theta)}{P(y|H_0)}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left\{\exp\left[\frac{2B}{\sigma^2}\int_0^T y(t)A\cos(2\pi f_0 t + \theta)\,dt \right.\right. \\ &\qquad \left.\left. - \frac{2B}{2\sigma^2}\int_0^T A^2\cos^2(2\pi f_0 t + \theta)\,dt\right]\right\}d\theta \end{aligned} \tag{6.3}$$

The integral in the exponent involving $\cos^2(2\pi f_0 t + \theta)$ is

$$\begin{aligned} \int_0^T \cos^2(2\pi f_0 t + \theta)\,dt &= \frac{T}{2}\left[1 + \frac{1}{T}\int_0^T \cos(4\pi f_0 t + 2\theta)\,dt\right] \\ &= \frac{T}{2}\left[1 + \frac{1}{2\pi f_0 T}\sin 2\pi f_0 T\cos(2\pi f_0 T + \theta)\right] \approx \frac{T}{2} \end{aligned} \tag{6.4}$$

where the approximation is very good for $f_0 T$, the number of cycles per pulse, greater than about 3 or 4, which will be assumed to be the case. Therefore, since $\sigma^2 = (N_0/2)(B + B) = N_0 B$ the likelihood ratio is

$$\frac{\exp(-A^2T/2N_0)}{2\pi}\int_0^{2\pi}\left\{\exp\left[-\frac{2A}{N_0}\int_0^T y(t)\cos(2\pi f_0 t + \theta)\,dt\right]\right\}d\theta \tag{6.5}$$

The exponent in the integrand can be written as

$$\begin{aligned} &-\frac{2A}{N_0}\int_0^T y(t)\cos(2\pi f_0 t + \theta)\,dt \\ &= -\frac{2A}{N_0} I\cos\theta + \frac{2A}{N_0} Q\sin\theta \\ &I = \int_0^T y(t)\cos 2\pi f_0 t\,dt, \quad Q = \int_0^T y(t)\sin 2\pi f_0 t\,dt \end{aligned} \tag{6.6}$$

and (6.5) becomes, using (3.52)

$$\exp\left(-\frac{A^2T}{2N_0}\right)\int_0^{2\pi}\exp\left[-\frac{2A}{N_0}(I\cos\theta - Q\sin\theta)\right]\frac{d\theta}{2\pi}$$
$$=\exp\left(-\frac{A^2T}{2N_0}\right)I_0\left(\frac{2Az}{N_0}\right) \tag{6.7}$$

where $z=\sqrt{I^2+Q^2}$ and I_0 is the modified Bessel function of order zero.[†] The likelihood Ratio Test (LRT) is therefore: choose H_1 if

$$I_0\left(\frac{2Az}{N_0}\right) \geq e^{E/N_0}\eta \tag{6.8}$$

and choose the null hypothesis H_0 otherwise; in (6.8) the exponent has been written in terms of the signal energy $E = A^2T/2$. Thus the modified Bessel function I_0 is the optimum non-coherent detection characteristic.

In what follows only the Neyman–Pearson criterion shall be applied. Since $I_0(x)$ is a monotonically increasing function of x, the LRT (6.8) can be expressed in an equivalent form as: choose the alternative H_1 if either

$$z=\sqrt{I^2+Q^2}\geq\eta \quad \text{or} \quad z^2=I^2+Q^2\geq\eta^2 \tag{6.9}$$

where the threshold η (not necessarily the same as that in (6.8)) is determined by the specified false alarm probability. In what follows we choose the former case and deal here with z and η. In Chapter 10, which deals with this subject using generalized complex signals, identical results are obtained using square-law detection which demonstrates the exact equivalence of envelope and square-law detection in this case.

Before dealing with the determination of η for the non-coherent case, let us consider the generation of the statistic z. This can be accomplished by the system illustrated in Figure 6.2 which is customarily referred to as a quadrature receiver, in which the paths employing multiplication by $\cos 2\pi f_0 t$ and $\sin 2\pi f_0 t$ are the in-phase and quadrature channels, yielding I and Q respectively.

Figure 6.2 is reminiscent of the correlation detector of Figure 5.4. That is, as discussed in connection with that figure, the multiplication-and-integration processes in the in-phase and quadrature channels in Figure 6.2 are equivalent to filters $h(t)$ matched to $P_T(t)\cos 2\pi f_0 t$ and $P_T(t)\sin 2\pi f_0 t$ respectively. Furthermore, the squaring adding and square-root operations which follow amount to envelope detection. The statistic z can therefore be generated by the system illustrated in Figure 6.3. To show this, note that for input $y(t)$ the output at any arbitrary time t is

[†] There should be no confusion between the notation I_0, the zero-order modified Bessel function, and I or I_t which denotes the in-phase channel term.

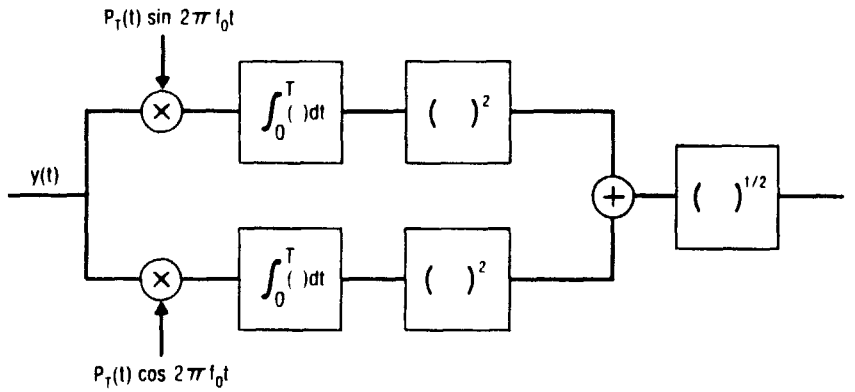


Figure 6.2 Quadrature receiver with envelope detection.

$$\int_0^t y(\tau)h(t-\tau)\,dt = \int_0^t y(\tau)\cos 2\pi f_0(T-t+\tau)\,d\tau$$
$$=\cos 2\pi f_0(T-t)\int_0^t y(\tau)\cos 2\pi f_0\tau\,d\tau$$
$$-\sin 2\pi f_0(T-t)\int_0^t y(\tau)\sin 2\pi f_0\tau\,d\tau$$

and the envelope is

$$\left[\left(\int_0^t y(\tau)\cos 2\pi f_0\tau\,d\tau\right)^2+\left(\int_0^t y(\tau)\sin 2\pi f_0\tau\,d\tau\right)^2\right]^{1/2}$$

which at the time of correlation, $t=T$, is equal to

$$\left[\left(\int_0^T y(\tau)\cos 2\pi f_0\tau\,d\tau\right)^2+\left(\int_0^T y(\tau)\sin 2\pi f_0\tau\,d\tau\right)^2\right]^{1/2}=\sqrt{I^2+Q^2}=z \tag{6.10}$$

As is discussed in Chapter 1, the envelope detector is also referred to as a linear detector, which, as illustrated in Figure 6.4, refers to that property of the detector whereby the output is proportional to the input when the input is positive. The operation of the detector however is of course highly

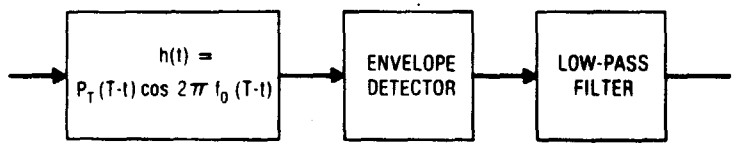


Figure 6.3 Equivalent of quadrature-receiver.

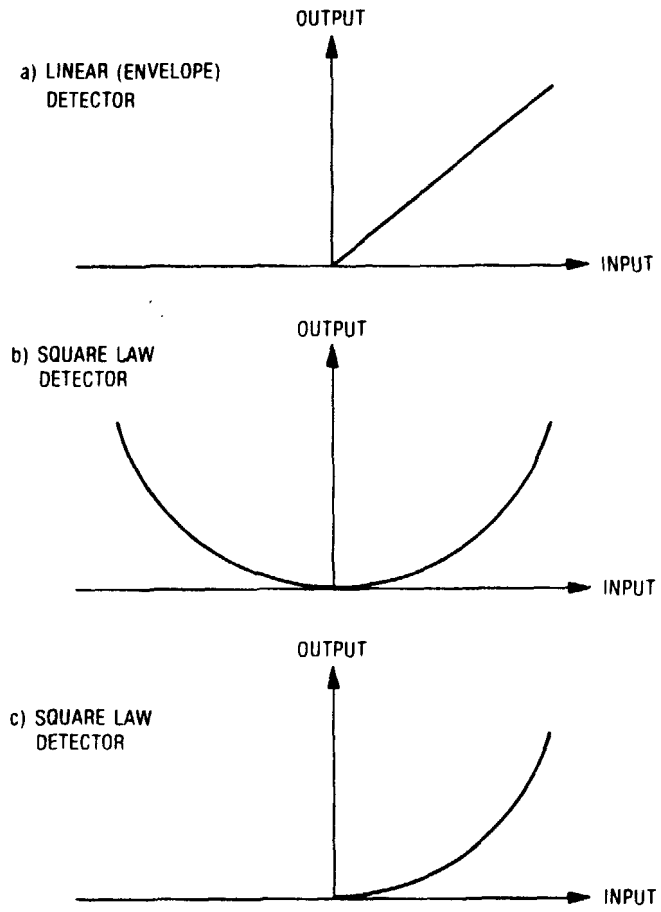


Figure 6.4 Various detector characteristics.

non-linear, as a result of which the output consists of a DC term equal to the envelope $\sqrt{I^2 + Q^2}$, the carrier term, plus an infinite number of harmonics of the input at $2f_0$, $3f_0$, etc. It is for this reason that, as discussed in Chapter 1, the output of the envelope detector must be passed through a low-pass filter or a video amplifier, which will eliminate the carrier term and the unwanted harmonics. Similar comments apply to square-law detectors which generate z^2, whose characteristics are shown in Figure 6.4c and d.

Now the transfer function of the filter in Figure 6.3 is

$$H(f) = \int_0^T \cos 2\pi f_0(T-t)e^{-i2\pi ft}\,dt$$

$$= \frac{e^{-i\pi(f-f_0)T}}{2}\,\frac{\sin \pi(f-f_0)T}{\pi(f-f_0)} + \frac{e^{-i\pi(f+f_0)T}}{2}\,\frac{\sin \pi(f+f_0)T}{\pi(f+f_0)} \qquad (6.11)$$

which is essentially that of a bandpass filter of bandwidth $\sim 1/T$ with the indicated phase characteristic, which of course is irrelevant here since the filtering is followed by envelope detection. Hence, for non-coherent detection the filtering operation is in practice implemented by a bandpass filter, with essentially arbitrary phase characteristics, whose bandwidth is nominally equal to that of the signal. As is discussed in [20], the purpose of bandwidth matching is to maximize SNR. Suppose the bandwidth is very narrow. If it is gradually widened, both the signal and the noise energy at the output will increase, but the signal energy will increase faster because it builds up coherently over its duration whereas the noise contributions at successive time instants are independent. Thus SNR increases. When the bandwidth is nominally equal to that of the signal however, further widening leaves the signal energy at the output unchanged and serves only to increase the noise, and SNR decreases. Hence the optimum is achieved when the filter bandwidth equals the signal bandwidth.

The filter bandwidth in this case is therefore matched to the signal bandwidth. If one wishes to be more careful the actual spectral shape of the filter can be matched to that of the signal. However, the matching is in amplitude only, and we do not actually have a matched filter since, as is discussed in connection with Figure 5.5, true matched filtering is equivalent to having available an *exact* replica of the signal, which includes the phase.

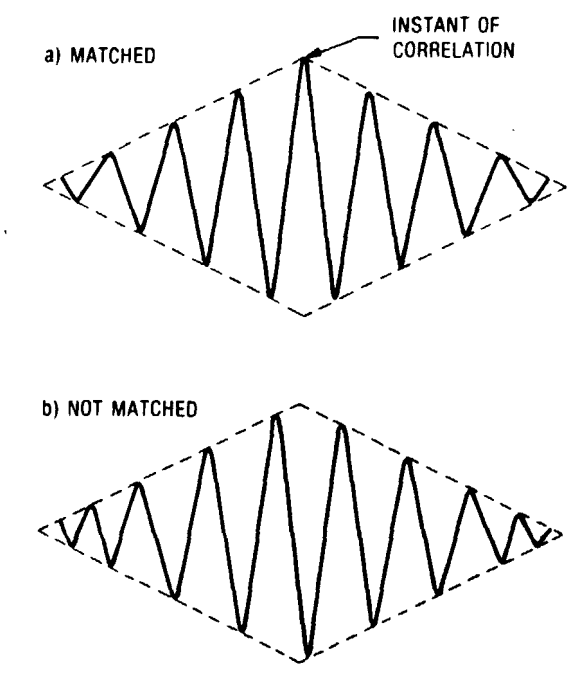


Figure 6.5 Outputs of matched and unmatched filters.

By the same token therefore, matched filtering is exactly equivalent to coherent detection and as noted, the results of Chapter 5 apply; one might denote the operation in this section as incoherent matched filtering [21]. To illustrate the difference between the two cases consider the outputs of: (a) a filter matched to a signal $P_T(t)A\cos(2\pi f_0 t + \theta)$ and (b) that of a filter whose impulse response is $P_T(t)A\sin 2\pi f_0(T - t)$, which are shown in Figure 6.5 (see Exercise 6.3). For the matched filter the instant of correlation (i.e. $t = T$) occurs simultaneously with a crest of the carrier yielding the maximum peak response. In the non-coherent case this will generally not be the case and the peak response may be smaller.

6.2 COMPARISON OF COHERENT AND NONCOHERENT DETECTION OF A SINGLE PULSE

We are now in a position to compare quantitatively the effectiveness of coherent and noncoherent detection of a single pulse. Referring to (6.6), the in-phase and quadrature components of the output of the bandpass filter in Figure 6.3 at $t = T$ are

$$I = \int_0^T y(t)\cos 2\pi f_0 t\,dt = \int_0^T A\cos(2\pi f_0 t + \theta)\cos 2\pi f_0 t\,dt$$

$$+ \int_0^T n(t)\cos 2\pi f_0 t\,dt = \frac{AT}{2}\cos\theta + \int_0^T n(t)\cos 2\pi f_0 t\,dt$$

$$Q = \frac{AT\sin\theta}{2} + \int_0^T n(t)\sin 2\pi f_0 t\,dt \tag{6.12}$$

where the approximation of (6.4) has been used. The quantities

$$X = \int_0^T n(t)\cos 2\pi f_0 t\,dt = I - \frac{AT\cos\theta}{2} \tag{6.13}$$

$$Y = \int_0^T n(t)\sin 2\pi f_0 t\,dt = Q - \frac{AT\sin\theta}{2}$$

are each mean-zero Gaussian random variables with variances

$$E(X^2) = E\int_0^T\int_0^T n(s)n(t)\cos 2\pi f_0 t\cos 2\pi f_0 s\,dt\,ds$$

$$E(Y^2) = E\int_0^T\int_0^T n(s)n(t)\sin 2\pi f_0 t\cos 2\pi f_0 s\,dt\,ds$$

and since for white noise

$$E(n(s)n(t)) = \frac{N_0}{2}\delta(s - t)$$

then to a very good approximation, again using (6.4)

$$E(X^2) = \frac{N_0}{2}\int_0^T \cos^2 2\pi f_0 t\,dt = \frac{N_0 T}{4} + \frac{N_0}{2}\int_0^T \cos 4\pi f_0 t\,dt$$

$$\approx \frac{N_0 T}{4} = \sigma_z^2 \tag{6.14}$$

with the same result for $E(Y^2)$. Therefore, since X and Y are independent Gaussian random variables their joint density is

$$P(X, Y)\,dX\,dY = \exp\left[-\frac{(X^2 + Y^2)}{2\sigma_z^2}\right]\frac{dX\,dY}{2\pi\sigma_z^2}$$

$$= \exp\left[-\frac{I^2 + Q^2 + (AT/2)^2 - AT(I\cos\theta + Q\sin\theta)}{2\sigma_z^2}\right]\frac{dX\,dY}{2\pi\sigma_z^2} \tag{6.15}$$

Now let $I = z\cos\phi$, $Q = z\sin\phi$, with $dX\,dY = dI\,dQ = z\,dz\,d\phi$. The exponent in (6.15) is then

$$-\frac{z^2 + (AT/2)^2 - zAT\cos(\theta - \phi)}{2\sigma_z^2} \tag{6.16}$$

and since $P(X, Y)\,dX\,dY = P(z, \phi)\,dz\,d\phi$, the probability density of the envelope z under H_1 is

$$P(z) = \int_0^{2\pi} P(z, \phi)\,d\phi = z\,\frac{\exp\{-[z^2 + (AT/2)^2]/2\sigma_z^2\}}{\sigma_z^2}$$

$$\times \frac{1}{2\pi}\int_0^{2\pi}\exp\left[-\frac{zAT}{2\sigma_z^2}\cos(\theta - \phi)\right]d\phi$$

$$= z\,\frac{\exp\{-[z^2 + (AT/2)^2]/2\sigma_z^2\}}{\sigma_z^2}\,I_0\left(\frac{zAT}{2\sigma_z^2}\right) \tag{6.17}$$

which is the Rice distribution that was introduced in Section 3.6, with a minor change in the argument (see Exercise 6.5).

Now under H_0, $A = 0$, and since $I_0(0) = 1$ (6.17) becomes under the null hypothesis the probability density for the envelope of noise alone, the Rayleigh distribution

$$P(z) = \frac{z}{\sigma_z^2}\exp\left(-\frac{z^2}{2\sigma_z^2}\right) \tag{6.18}$$

The threshold η in (6.8) is therefore determined from the false alarm probability P_{fa} using

$$P_{fa} = \int_\eta^\infty z\,\frac{\exp(-z^2/2\sigma_z^2)}{\sigma_z^2}\,dz = e^{-\eta^2/2\sigma_z^2} \tag{6.19}$$

whence, using (6.14)

$$\eta = (-2\sigma_z^2 \log P_{fa})^{1/2} = \left(-\frac{N_0 T}{2} \ln P_{fa}\right)^{1/2} \tag{6.20}$$

and the detection probability P_d is

$$P_d = \int_\eta^\infty z \frac{\exp[-(z^2 + (AT/2)^2)/2\sigma_z^2]}{\sigma_z^2} I_0\left(\frac{zAT}{2\sigma_z^2}\right) dz$$

This can be put into a more convenient dimensionless form with the change of variables $x = z/\sigma_z$ from which

$$P_d = \int_{(-2 \ln P_{fa})^{1/2}}^\infty x \exp\left[-\frac{(x^2 + \alpha^2)}{2}\right] I_0(x\alpha)\, dx \tag{6.21}$$

where $\alpha^2 = 2E/N_0$.

The expression in (6.21) is known as Marcum's Q function [22], and shows explicitly that detection probability depends only on P_{fa} and on the signal-to-noise ratio E/N_0. For coherent detection the results for the matched filter—(5.52) et seq.—are

$$P_d = \frac{1}{2}\left[1 + \operatorname{erf}\left(\sqrt{\frac{E}{N_0}} - \gamma\right)\right] \tag{6.22}$$

where the numerical parameter γ is determined from P_{fa} through the relationship

$$P_{fa} = \tfrac{1}{2}[1 - \operatorname{erf}(\gamma)] \tag{6.23}$$

and the detection threshold η is

$$\eta = 2\gamma\sqrt{\frac{E}{N_0}} \tag{6.24}$$

Note that, in addition to P_{fa}, the detection threshold depends on the received signal energy in the coherent case (6.23), but only on the noise and the integration time in the non-coherent case (6.20), which is somewhat more convenient.

A comparison of coherent and non-coherent detection using (6.21), (6.22) and (6.23) is presented in Figure 6.6 in terms of the value of P_d that can be achieved for a given value of P_{fa} as a function of signal-to-noise ratio E/N_0. It is seen that: (1) for small values of E/N_0—say $E_0 < 3$ dB—for any given value of P_{fa}, noncoherent detection requires from 2 to 3 dB more SNR than that required by coherent detection in order to achieve the same value of P_d; (2) for large values of E/N_0—say >10 dB—the difference in SNR

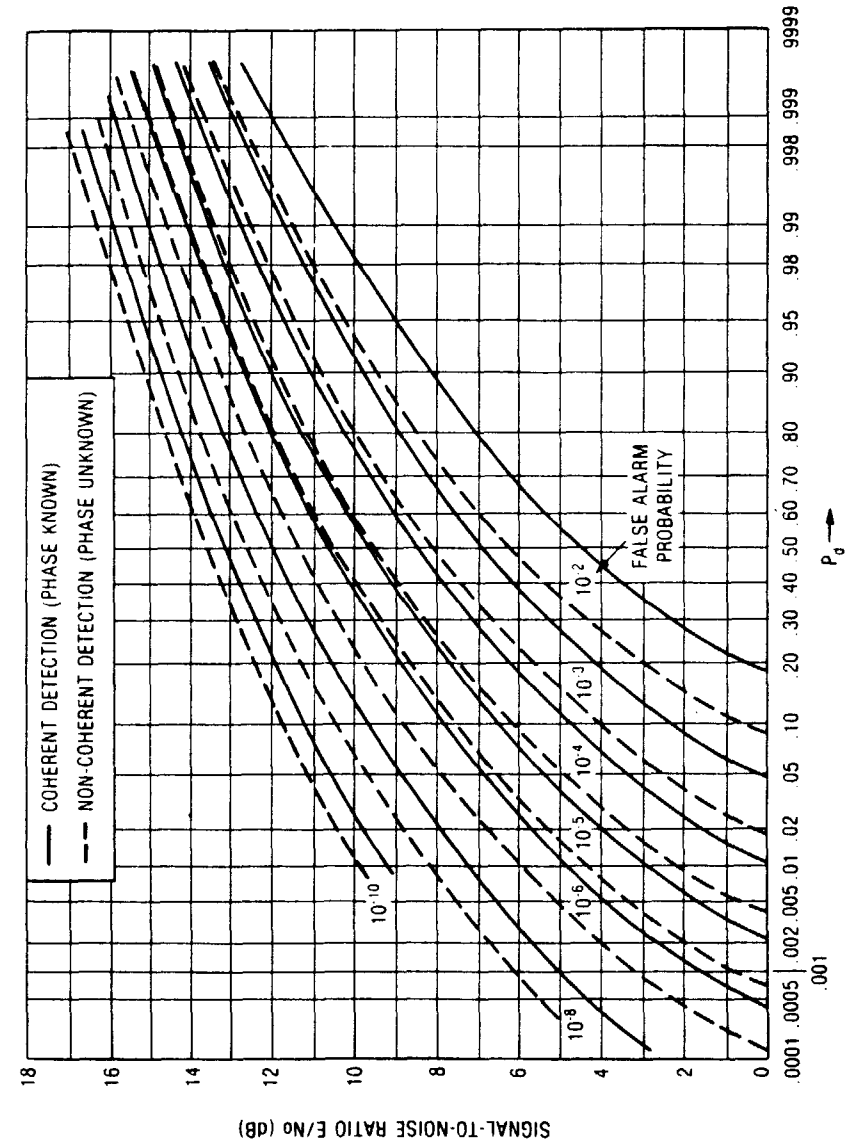


Figure 6.6 Detection probability for coherent and non-coherent detection of a single pulse (adapted from [21]).

required by the two schemes is less than ~1 dB, and it is clear that with further increase E/N_0 the difference eventually becomes negligible.

These results may be understood as follows. With a coherent matched filter the instantaneous sinusoidal amplitude of the filter output at some time t is

$$\left[\frac{At}{2} + \int_0^t n(\tau)\cos 2\pi f_0\tau\, d\tau\right]\cos 2\pi f_0(T-t) - \left[\int_0^t n(\tau)\sin 2\pi f_0\tau\, d\tau\right]\sin 2\pi f_0(T-t) \tag{6.25}$$

where, again, (6.4) has been used and θ has been set equal to zero for convenience. In contrast with the non-coherent case (e.g. (6.12)), the signal component is contained entirely within the in-phase channel and, referring to (3.45), therefore contends with only the in-phase noise component; the quadrature channel need not come into play. Now the signal component at the matched filter output at the instant of correlation is $AT/2$ and from (6.14), $E(X^2) = E(Y^2) = N_0T/4$. Hence SNR for a matched filter is therefore $A^2T^2/N_0T = A^2T/N_0 = 2E/N_0$.

On the other hand, in the non-coherent case both the in-phase and quadrature noise components come into play, and in a similar calculation the noise term would be $N_0T/2$. This is clear from the Rayleigh distribution for the envelope of noise alone, for which $E(r^2) = 2\sigma^2$. Hence, if the signal amplitude A is not very large relative to σ, both noise components affect the random fluctuations in the envelope equally, and SNR is just E/N_0. However, if A is relatively large the fluctuations in the envelope of signal plus noise will be caused for the most part only by those noise components in phase with the signal and the signal-to-noise ratio is therefore effectively increased by 3 dB. To show this,[†] write the envelope of (6.24) as:

$$[(A+n_c)^2 + n_s^2]^{1/2} = A\left(1 + \frac{2n_c}{A} + \frac{n_c^2+n_s^2}{A^2}\right)^{1/2} \approx A + n_c + \frac{n_c^2+n_s^2}{2A} \approx A + n_c \tag{6.26}$$

if A is large. Hence the fluctuations in the observable are carried primarily by n_c, and since $E(n_c^2) = \sigma^2 = N_0B = N_0/T$, then:

$$\frac{A^2}{\sigma^2} = \frac{A^2T}{N_0} = \frac{2E}{N_0} \tag{6.27}$$

From still another point of view, let us also recall (see Exercise 3.10) that if E/N_0 is large the Rice distribution can be approximated as a Gaussian,

[†] This can also be illustrated by a simple sketch based on Figure 3.9b.

with exponent $(r-A)^2/2\sigma^2$ and $A^2/\sigma^2 = 2E/N_0$. So to summarize, with a coherently matched filter, for which the input signal phase must be known exactly, an advantage over non-coherent detection is achieved when E/N_0 is small, which is equivalent to an improvement in E/N_0 by about a factor of two which is seen in Figure 6.6. On the other hand, if E/N_0 is large, say >10 dB, there is little difference between coherent and non-coherent matched filtering for a single pulse.

6.3 IMPROVEMENT IN SIGNAL-TO-NOISE RATIO BY COHERENT AND NONCOHERENT INTEGRATION

In the previous section coherent and noncoherent detection have been compared when the signal consists of a single pulse. In such cases the sensitivity of the sensor—that is, the maximum range at which a target can be reliably detected—is limited by the maximum available energy per pulse. The sensitivity can be significantly increased however, with no increase in peak power, by transmitting waveforms consisting of a large number of pulses and adding up their cumulative effect; this technique is also used to improve SNR in communication systems. In this case the average power P_TT/T_p, where T_p is the interpulse spacing, and P_T is the transmitted signal power per pulse, must be kept within the capability of the transmitter—the quantity T/T_p is known as the duty factor. As has been noted, there are two approaches to this type of signal processing, known as non-coherent integration and coherent integration, which will now be evaluated and compared in terms of the improvement in SNR that is achieved in the two cases.

6.3.1 Noncoherent Integration

The noncoherent case shall be considered first. Equation (6.7) gives the likelihood ratio for a single observation. It then follows at once that for M independent observations the likelihood ratio is

$$e^{-MA^2T/2N_0}\prod_{i=1}^{M} I_0\left(\frac{2Az_i}{N_0}\right) \tag{6.28}$$

where $z_i = \sqrt{I_i^2 + Q_i^2}$ is the ith output of the detector in Figure 6.3 and

$$I_i^2 = \left(\int_0^T y_i(\tau)\cos 2\pi f_0\tau\, d\tau\right)^2$$
$$Q_i^2 = \left(\int_0^T y_i(\tau)\sin 2\pi f_0\tau\, d\tau\right)^2 \tag{6.29}$$

where $y_i(t)$ is the ith observation, $i = 1, 2, \ldots, M$. The extension of the single-pulse LRT of (6.8) to a pulse-train waveform therefore becomes,

after taking logarithms, declare H_1 if

$$\sum_{i=1}^{M} \ln I_0\left(\frac{2Az_i}{N_0}\right) \geq \frac{MA^2T}{2N_0} + \ln \eta \tag{6.30}$$

Now since the statistic consists of a sum of terms instead of a single term as in (6.8), the monoticity argument that was used there can no longer be applied and implementation of the I_0 characteristic, which as noted above is optimum for noncoherent detection, must be considered. It has been shown in Section 6.2 that the difference between coherent and noncoherent detection of a single pulse becomes negligible when SNR is large. In the interest of this comparison therefore let us consider the case when $2Az_i/N_0$ is small, and then make use of the approximation for small x

$$I_0(x) \approx 1 + x^2$$

Thus

$$\ln I_0\left(\frac{2Az_i}{N_0}\right) \approx \ln\left[1 + \left(\frac{2Az_i}{N_0}\right)^2\right] \approx \left(\frac{2Az_i}{N_0}\right)^2$$

and the I_0 characteristic is implemented by a square-law detector. Equation (6.30) therefore becomes

$$\sum_{i=1}^{M} z_i^2 \geq \frac{MTN_0}{8} + \frac{N_0^2}{4A^2} \ln \eta \tag{6.31}$$

The process of summing successive outputs after detection is illustrated in Figure 6.7. The integration is noncoherent because the phase information which contains the phase relationship between successive pulses is destroyed in the detection process.

For completeness, for large values of $2Az_i/N_0$, $I_0(2Az_i/N_0)$ is approximated by

$$I_0\left(\frac{2Az_i}{N_0}\right) \approx \frac{\exp(2Az_i/N_0)}{(4\pi Az_i/N_0)^{1/2}}$$

so that

$$\ln I_0\left(\frac{2Az_i}{N_0}\right) \approx \frac{2Az_i}{N_0} - \frac{1}{2}\ln\left(\frac{4\pi Az_i}{N_0}\right) \approx \frac{2Az_i}{N_0}$$

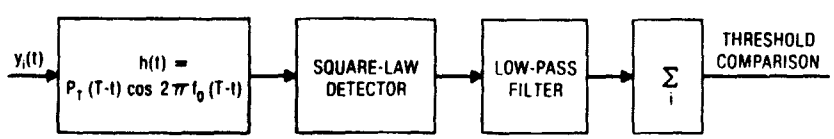


Figure 6.7 System for post-detection integration.

and (6.31) becomes, choose H_1 if

$$\sum_{i=1}^{M} z_i \geq \frac{MAT}{4} + \frac{N_0}{2A} \ln \eta \tag{6.32}$$

In this case the square-law detector in Figure 6.7 would be replaced by a linear, or envelope, detector. Thus, the rule for optimal implementation of the ideal Bessel function characteristic is to use linear detection for large SNR and square-law detection for small SNR. However there is little difference in performance for most cases of interest [21].

Continuing with square-law detection, we now introduce the deflection signal-to-noise ratio. If one were observing the output of a nonlinear detector on an oscilloscope, the appearance of the trace with the arrival of a signal at time t_0 would typically be as illustrated in Figure 6.8. Although, prior to t_0 the detector input consists of mean-zero Gaussian noise, because of the nonlinear operation the mean value at the output is no longer zero, which has been shown. Thus, prior to the time t_0, the trace a_n for noise only has some nonzero average value $\overline{a_n}$, and rms fluctuation

$$\sigma_a = (\overline{a_n^2} - \bar{a}_n^2)^{1/2} \tag{6.33}$$

and with the arrival of the signal the trace would be deflected to a value

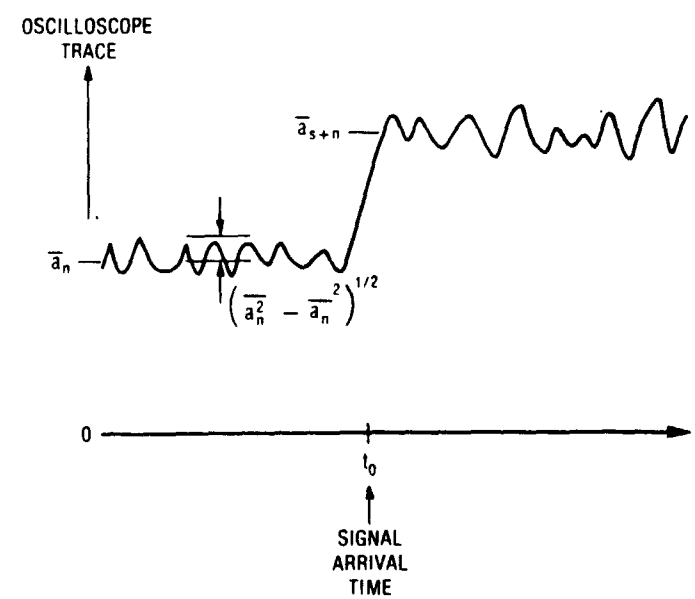


Figure 6.8 Output of non-linear detector.

a_{s+n}. For an observer who is looking for the deflection of the trace as an indication of the presence of a target (or a digital signal pulse) the quantity

$$\frac{\overline{a_{s+n}} - \overline{a_n}}{\sigma_a}$$

is a reasonable measure of the extent to which the target's presence would be detected. For example, a value of 10 would mean that the average deflection from $\overline{a_n}$ due to the presence of a target is ten-times the rms fluctuations of the trace due to noise.

In the case of interest here the observed quantity is the result of integration of M successive outputs of the square-law detector, given by $Z = \Sigma_{i=1}^{M} z^2(i)$ and the deflection signal-to-noise ratio DSNR [23] is defined here as

$$\text{DSNR} = \frac{\bar{Z}_{s+n} - \bar{Z}_n}{\sigma_Z} \tag{6.34}$$

where $\sigma_Z = (\overline{Z_n^2} - \bar{Z}_n^2)^{1/2}$. The quantity Z_{s+n} involves summation of square-law-detector outputs $z_{s+n}^2(i)$ when signal and noise are present, and Z_n denotes noise only. We now calculate DSNR. The ith observation $y_i(t)$ is

$$y_i(t) = AP_T(t)\cos(2\pi f_0 t + \theta_i) + n_i(t)$$

with the usual assumptions for $n(t)$, and using (6.10)

$$z_{s+n}^2(i) = \left[\int_0^T [A\cos(2\pi f_0\tau + \theta_i) + n_i(\tau)]\cos 2\pi f_0\tau\, d\tau\right]^2 + \left[\int_0^T [A\cos(2\pi f_0\tau + \theta_i) + n_i(\tau)]\sin 2\pi f_0\tau\, d\tau\right]^2 = I_i^2 + Q_i^2 \tag{6.35}$$

where in the interest of maximum generality the unknown signal phase θ_i is allowed to vary randomly and independently over $(0, 2\pi)$ between successive observations. The terms I_i and Q_i are explicitly

$$I_i = A\cos\theta_i \int_0^T \cos 2\pi^2 f_0\tau\, d\tau - A\sin\theta_i \int_0^T \sin 2\pi f_0\tau \cos 2\pi f_0\tau\, d\tau + \int_0^T n_i(\tau)\cos 2\pi f_0\tau\, d\tau$$

$$Q_i = A\cos\theta_i \int_0^T \cos 2\pi f_0\tau \sin 2\pi f_0\tau\, d\tau - A\sin\theta_i \int_0^T \sin 2\pi f^2 f_0\tau\, d\tau + \int_0^T n_i(\tau)\sin 2\pi f_0\tau\, d\tau \tag{6.36}$$

For the same reasons discussed in connection with (6.4) the integrals with integrands $\sin 2\pi f_0\tau \cos 2\pi f_0\tau$ are negligible, and also

$$\int_0^T \sin^2 2\pi f_0\tau\, d\tau = \int_0^T \cos^2 2\pi f_0\tau\, d\tau \approx \frac{T}{2}$$

Hence, as before, using the notation

$$X_i = \int_0^T n_i(\tau)\cos 2\pi f_0\tau\, d\tau$$

$$Y_i = \int_0^T n_i(\tau)\sin 2\pi f_0\tau\, d\tau \tag{6.37}$$

the ith output of the square-law detector when the signal is present is

$$z_{s+n}^2(i) = \left(\frac{AT}{2}\cos\theta_i + X_i\right)^2 + \left(\frac{AT}{2}\sin\theta_i + Y_i\right)^2 = \frac{A^2T^2}{4} + AT(X_i\sin\theta_i + Y_i\cos\theta_i) + X_i^2 + Y_i^2 \tag{6.38}$$

X_i and Y_i have the following properties. Referring to (6.14) et seq.

$$E(X_i^2) = E(Y_i^2) = \frac{N_0T}{4} \tag{6.39}$$

Also by using the identity for Gaussian random variables ξ_i

$$E(\xi_1\xi_2\xi_3\xi_4) = E(\xi_1\xi_2)E(\xi_3\xi_4) + E(\xi_1\xi_3)E(\xi_2\xi_4) + E(\xi_1\xi_4)E(\xi_2\xi_3) \tag{6.40}$$

it is easily shown that (see Exercise 6.6).

$$E(X_i^4) = E(Y_i^4) = 3\frac{N_0^2T^2}{16} \tag{6.41}$$

and

$$E(X_i^2Y_i^2) = E(X_i^2)E(Y_i^2) = \frac{N_0^2T^2}{16} \tag{6.42}$$

That is, X_i^2 and Y_i^2 are uncorrelated. And by setting $A = 0$ in (6.38)

$$E[z_n^2(i)] = E(X_i^2 + Y_i^2) = \frac{N_0T}{2} \tag{6.43}$$

With the use of these results, the numerator of (6.34) is

$$\bar{Z}_{s+n} - \bar{Z}_n = E\left[\sum_{i=1}^{M} z_{s+n}^2(i) - \sum_{i=1}^{M} z_n^2(i)\right] = \frac{MA^2T^2}{4} \tag{6.44}$$

The square of the denominator σ_Z^2 is

$$\begin{aligned}\sigma_Z^2 &= E(Z_n^2) - [E(Z_n)]^2 \\ &= E\left[\left(\sum_{i=1}^{M} X_i^2 + Y_i^2\right)^2\right] - \left[E\sum_{i=1}^{M}(X_i^2 + Y_i^2)\right]^2 \qquad (6.45)\\ &= E\sum_i (X_i^2 + Y_i^2)\sum_j (X_j^2 + Y_j^2) - M^2\left(\frac{N_0T}{2}\right)^2 \qquad (6.46)\end{aligned}$$

where (6.39) has been used, and the summation terms reduce to

$$\begin{aligned}&\sum_i (X_i^2 + Y_i^2)\sum_j (X_j^2 + Y_j^2)\\ &= \sum_i X_i^4 + \sum_i\sum_j X_i^2X_j^2 + \sum_i Y_i^4 \quad (i \neq j)\\ &\quad + \sum_i\sum_j Y_i^2Y_j^2 + 2\sum_i X_i^2 \sum_j Y_j^2 \quad (i \neq j)\end{aligned} \tag{6.47}$$

In the double-summations in which $i \neq j$ there are $M(M-1)$ terms, in which X_i^2 and X_j^2 are independent, as are Y_i^2 and Y_j^2. Also by (6.42) X_i^2 and Y_i^2 are independent for all i. Equation (6.47) then becomes, using (6.39)–(6.44):

$$E\sum_i (X_i^2 + Y_i^2)\sum_j (X_j^2 + Y_j^2) = \frac{MN_0^2T^2}{4} + \frac{M^2N_0^2T^2}{4} \tag{6.48}$$

and using (6.46)

$$\sigma_Z^2 = \frac{MN_0^2T^2}{4} \tag{6.49}$$

from which, using (6.44), (6.34) is

$$\text{DSNR} = \frac{MA^2T^2/4}{(MN_0^2T^2/4)^{1/2}} = \sqrt{M}\,\frac{A^2T}{2N_0} = \sqrt{M}\,\frac{E}{N_0} \tag{6.50}$$

where $E = A^2T/2$ in the signal energy. Since in this scheme the integration takes place at the output of the detector, noncoherent integration is also referred to as post-detection integration, which yields an improvement in DSNR by a factor equal to the square root of the number of summations. For a linear detector it can be shown that the improvement also goes as $\sqrt{M}$. This case however is more cumbersome and is not worked out here.

6.3.2 Coherent Integration

According to [23] the first published report proposing the use of coherent integration was by Emslie [24]. In essence, the processor maintains the phase relationship between the received pulses so that the signal amplitude adds coherently. Since the noise is independent from pulse to pulse an improvement in effective signal-to-noise ratio over noncoherent integration can be achieved. The outputs of the integrator are usually input to, say, a square-law detector, and coherent integration is therefore also referred to as predetection integration. As has been noted, it is not necessary for the exact value of the signal phase to be known for this purpose. If this value is known some additional benefit can be obtained, exactly analogous to that obtained by coherent over noncoherent detection of a single pulse discussed in Section 6.2. In fact, it will be shown that this alternative is equivalent to implementing a filter that is matched to the entire pulse train, and in what follows we make the distinction between predetection integration, or coherent integration, and what might be termed generalized matched filtering. It will also be shown however that, practically speaking, there would be no significant advantage in doing this if increase in receiver sensitivity were the only purpose.

A receiver for implementing coherent integration is illustrated in Fig. 6.9. The input is a train of M pulses, each one of the form $P_T(t)\cos(2\pi f_0 t + \theta)$, where θ is assumed to be unknown. This might represent backscatter from a target or a repeated "1" in a binary communication channel. The oscillator used in generating the transmitted signal as well as the local oscillators in the receiver I and Q channels would ordinarily be derived from the same clock. Coherent operation requires that the clock maintain phase stability over time periods of the order of the duration of the M-pulse waveform or the round-trip travel time up to and back from the target, whichever is shorter.[†]

Noise is not shown in the figure, nor are the sum-frequency terms at the outputs of the multipliers, which would be eliminated by the IF stage. Also, the echo would in general have undergone a Doppler shift due to target motion that would have to be accommodated. This could be accomplished by employing a bank of such receivers, in which contiguous local oscillators would be offset in frequency by an amount equal to the width of a Doppler-resolution cell. In this way the entire range of possible Doppler shifts could be covered, and the range rate of the target determined by noting which of the receivers produces the maximum output. Alternatively, Doppler shifts could be accommodated by employing a large-BT Doppler-invariant waveform such as is described in Chapters 8 and 9. For purposes of simplicity in this presentation however, target motion is not considered here.

[†] Practically speaking, it is actually only necessary that the two local oscillator signals remain orthogonal over this time period.

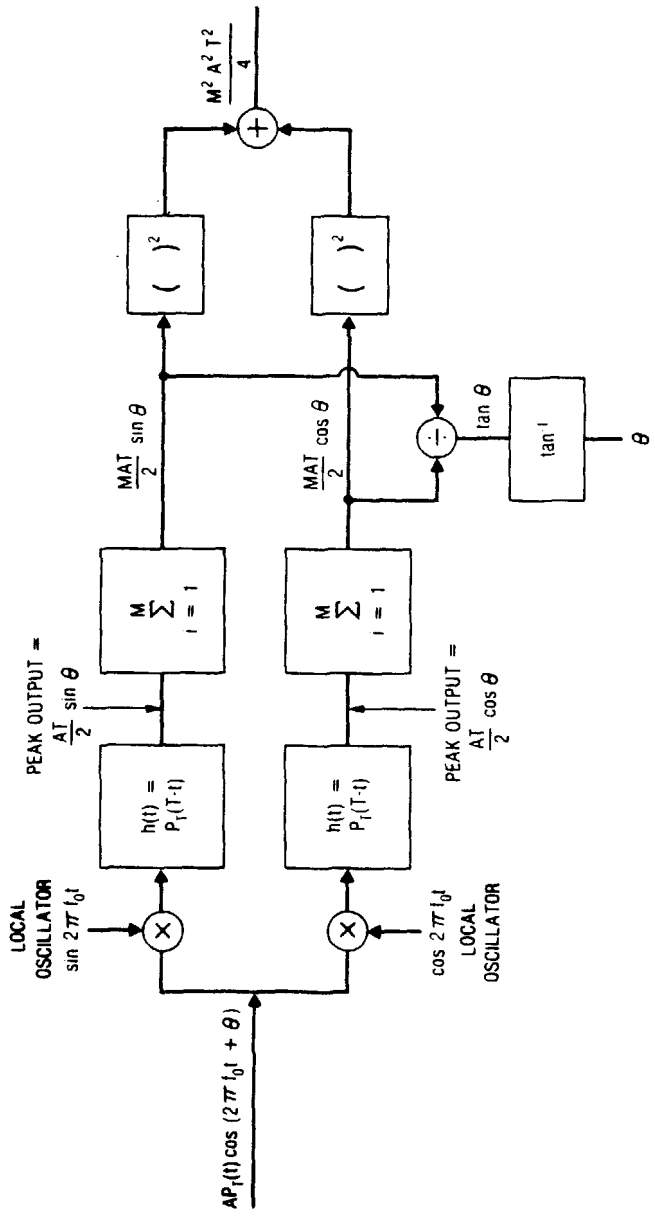


Figure 6.9 Receiver for coherent integration.

The outputs of the multipliers are passed through baseband filters matched to the video pulse shape, and the peak outputs of the filters in the in-phase and quadrature channels are $AT\cos\theta/2$ and $AT\sin\theta/2$ respectively. The integration which follows, by which SNR improvement is achieved, is accomplished by periodically overlapping and adding successive filter responses, the period being equal to the time separation between successive pulse transmissions. At this point the phase can be determined, if so desired, by dividing the quadrature term by the in-phase term and applying the arc-tangent operation. However, it is only necessary to add the outputs of the square-law detectors to acquire the coherently summed signal amplitude; measurement of θ is not necessary for this purpose. This is possible because $\sin^2\theta + \cos^2\theta = 1$ and a quadrature receiver must therefore be used. Eliminating the integrator in Figure 6.9—or equivalently setting $M = 1$—yields a receiver for measuring the phase of a single pulse, on the basis of which a coherent matched filter such as is discussed in Section 6.2 could be implemented.

In Figure 6.9 it has been tacitly assumed that the unknown phase θ remains constant over the duration of the M-pulse waveform, which (aside from deterministic variation due to target motion which we do not consider here) is generally assumed to be the case when coherent integration is employed. This assumption was not necessary in the preceding section for noncoherent integration. Here, however, it is clear that if this is not the case the signal will not add coherently from pulse to pulse and the process will not yield the desired SNR improvement.

Now, referring to Figure 6.9 and (6.13), at the instant of correlation for the ith signal pulse the outputs I_i and Q_i of the filters in the in-phase and quadrature channels are

$$\begin{aligned} I_i &= \frac{AT\cos\theta}{2} + \int_0^T n_i(\tau)\cos 2\pi f_0\tau\, d\tau \\ &= \frac{AT\cos\theta}{2} + X_i \\ Q_i &= \frac{AT\sin\theta}{2} + \int_0^T n_i(\tau)\sin 2\pi f_0\tau\, d\tau \\ &= \frac{AT\sin\theta}{2} + Y_i \end{aligned} \tag{6.51}$$

which becomes, after integration

$$\begin{aligned} \sum_{i=1}^{M} I_i &= \frac{MAT\cos\theta}{2} + \sum_{i=1}^{M} X_i \\ \sum_{i=1}^{M} Q_i &= \frac{MAT\sin\theta}{2} + \sum_{i=1}^{M} Y_i \end{aligned} \tag{6.52}$$

And after adding the outputs of the square-law detectors for the in-phase and quadrature channels

$$Z_{s+n} = \left(\frac{MAT\cos\theta}{2} + \sum_{i=1}^{M} X_i\right)^2 + \left(\frac{MAT\sin\theta}{2} + \sum_{i=1}^{M} Y_i\right)^2 \tag{6.53}$$

In order to calculate DSNR it is necessary to evaluate $E(Z_{s+n}^2)$, which is equal to

$$E(Z_{s+n}^2) = \frac{M^2A^2T^2}{4} + E\left[\left(\sum_i X_i\right)^2 + \left(\sum_i Y_i\right)^2\right]$$

because I_i and Q_i have zero mean, and

$$\overline{Z_{s+n}^2} - \overline{Z_n^2} = \frac{M^2A^2T^2}{4} \tag{6.54}$$

For the denominator of (6.34) we must calculate the square root of the quantity

$$E\left[\left(\sum_i X_i\right)^2 + \left(\sum_i Y_i\right)^2\right]^2 - \left[E\left(\sum_i X_i\right)^2 + E\left(\sum_i Y_i\right)^2\right]^2 \tag{6.55}$$

The first term is

$$E\left[\left(\sum_i X_i\right)^4 + \left(\sum_i Y_i\right)^4 + 2\left(\sum_i X_i\right)^2\left(\sum_i Y_i\right)^2\right] \tag{6.56}$$

By writing

$$\left(\sum_i X_i\right)^4 = \sum_i X_i \sum_j X_j \sum_k X_k \sum_l X_l \tag{6.57}$$

and using (6.40) it is easily shown that

$$\left(\sum_{i=1}^{M} X_i\right)^4 + \left(\sum_{i=1}^{M} Y_i\right)^4 = \frac{6M^2N_0^2T^2}{16} \tag{6.58}$$

The cross term in (6.56) is, using (6.42),

$$2E\left(\sum_i X_i\right)^2\left(\sum_i Y_i\right)^2 = 2E\left(\sum_i X_i^2 \sum_i Y_i^2\right) = \frac{2M^2N_0^2T^2}{16} \tag{6.59}$$

so that (6.56) is equal to $M^2N_0^2T^2/2$. Hence (6.55) is

$$\frac{M^2N_0^2T^2}{2} - \frac{M^2N_0^2T^2}{4} = \frac{M^2N_0^2T^2}{4} \tag{6.60}$$

and finally, from (6.34)

$$\text{DSNR} = \frac{M^2A^2T/4}{(M^2N_0^2T^2/4)^{1/2}} = \frac{MA^2T}{2N_0} = M\frac{E}{N_0} \tag{6.61}$$

Thus we have the important result that the improvement in SNR yielded by coherent integration is equal to the number of coherent additions M, whereas as seen in (6.50) for noncoherent or postdetection integration the improvement goes as $\sqrt{M}$, the square-root of the number of terms in the integration. But M is also the increase over a single pulse of the received signal energy. Thus the essential feature of coherent processing is exhibited; namely, that all the energy in the received signal is effectively recovered.

To achieve this it was only necessary to be differentially coherent from pulse to pulse. Now suppose θ were known, and consider a filter matched to each pulse, with impulse response: $h(t) = P_T(T-t)\cos(2\pi f_0(T-t)+\theta)$. It is not difficult (see Exercise 6.7) to show that in this case, in comparison with (6.51), the ith output of the in-phase channel at $t = T$ is $\frac{1}{2}AT + X_i$. Thus the signal is confined entirely within the in-phase channel, the quadrature channel need not be considered, and the integrated output is

$$\frac{MAT}{2} + \sum_{i=1}^{M} X_i \tag{6.62}$$

and using the definition of SNR of (5.61)

$$\text{SNR} = \frac{M^2A^2T^2/4}{E[\Sigma_{i=1}^{M} X_i]^2} = \frac{M^2A^2T^2}{MN_0T} = \frac{MA^2T}{N_0} = \frac{M2E}{N_0} \tag{6.63}$$

In fact, we have effectively implemented a generalized coherent matched filter to a waveform consisting of M repeated pulses, as evidenced by the factor of 2. The reason the integrated SNR in this case is twice that yielded by the quadrature detector, (6.61), is because the noise in the quadrature channel as well as the in-phase channel comes into play there. The situation is therefore an exact parallel to that discussed in Section 6.2, and it will be seen that the comparison in performance of coherent integration and generalized matched filtering also exactly parallels coherent and noncoherent detection of a single pulse.

6.4 PERFORMANCE OF COHERENT AND NONCOHERENT INTEGRATION

Let us now compare the performance of the integration schemes that have been considered, in terms of false alarm and detection probabilities.

6.4.1 Noncoherent Integration

Referring to (6.38), the output of the square-law detector can be written as

$$S_M = \sum_{i=1}^{M} \left(\frac{AT}{2}\cos\theta_i + X_i\right)^2 + \left(\frac{AT}{2}\sin\theta_i + Y_i\right)^2 \tag{6.64}$$

where X_i and Y_i are mean-zero Gaussian random variables with variance $N_0T/4$. The random variable S_M has a noncentral chi-square distribution with $2M$ degrees of freedom. For an exact solution the reader is referred to [21] which presents a concise and thorough discussion of chi-square together with exact performance calculations giving P_d vs P_f as a function of E/N_0 and M based on expansion of integrals of the chi-square distribution in a Gram–Charlier series. Here, a simplified approximate solution shall be presented which approaches the exact solution as M becomes large and in fact is accurate to 1 dB for $M \geq 4$ for typical situations of interest.

Equation (6.64) can be written as

$$S_M = \sum_{i=1}^{M} R_i \tag{6.65}$$

with

$$R_i = \left(\frac{AT}{2}\cos\theta_i + X_i\right)^2 + \left(\frac{AT}{2}\sin\theta_i + Y_i\right)^2$$

where the R_i are independent, identically distributed random variables. By the central-limit theorem therefore (Section 2.9), the probability density function of S_M approaches a Gaussian in the limit as $M \to \infty$ with mean μ_M and variance σ_M^2 given by

$$\mu_M = ME(R_i) = \frac{MA^2T^2}{4} + \frac{MN_0T}{2} = \frac{MN_0T}{2}\left(1 + \frac{E}{N_0}\right) \tag{6.66}$$

$$\sigma_M^2 = M[E(R_i^2) - (E(R_i))^2]$$

where the value of μ_M follows directly from (6.38). Calculation of σ_M^2 requires evaluating

$$\begin{aligned} E(R_i^2) &= E\left(\frac{A^2T^2}{4} + ATX_i\cos\theta_i + ATY_i\sin\theta_i + X_i^2 + Y_i^2\right)^2 \\ &= E(a+b+c+d+e)^2 \\ &= E(a^2+b^2+c^2+d^2+e^2+2ab+2ac+2ad+2ae \\ &\quad + 2bc+2bd+2be+2cd+2ce+2de) \end{aligned} \tag{6.67}$$

Equation (6.67) is simple to evaluate because the expected value of all cross terms involving odd powers of either X_i or Y_i vanish, and by using (6.39) and (6.41) (see Exercise 6.8),

$$\sigma_M^2 = \frac{MN_0^2T^2}{4}\left(1 + 2\frac{E}{N_0}\right) \tag{6.68}$$

Now for the false alarm probability set E equal to zero in (6.66) and (6.68), let $\mu = MN_0T/2$ and $\sigma^2 = MN_0^2T^2/4$, and the threshold η is determined from

$$P_{fa} = \int_\eta^\infty e^{-(x-\mu)^2/2\sigma^2}\frac{dx}{(2\pi\sigma^2)^{1/2}} = \frac{1}{2}(1 - \mathrm{erf}(\gamma)) \tag{6.69}$$

where

$$\gamma = \left(\frac{\mu - \eta}{\sqrt{2}\sigma}\right)$$

and

$$\eta = \frac{M}{2}N_0T\left(1 + \gamma\sqrt{\frac{2}{M}}\right) \tag{6.70}$$

The detection probability is

$$\begin{aligned} P_d &= \int_\eta^\infty e^{-(x-\mu_M)^2/2\sigma_M^2}\frac{dx}{\sqrt{2\pi}\sigma_M} \\ &= \frac{1}{2}\left\{1 + \mathrm{erf}\left[\frac{\left(\sqrt{\frac{M}{2}}\frac{E}{N_0} - \gamma\right)}{\left(1 + \frac{2E}{N_0}\right)^{1/2}}\right]\right\} \end{aligned} \tag{6.71}$$

which shows more directly the benefit in the improvement of SNR, and that P_d can be made arbitrarily close to unity, for any γ, if M can be made large enough.

A physical picture of the effect of M can be gained by noting that the separation of the peaks of the Gaussian distributions under H_0 and under H_1, is, from (6.66), equal to $ETM/2$. Also, the standard deviation under H_0 is $\sqrt{M}N_0T/2$. In order to achieve a low value of P_{fa} and a sufficiently large P_d it is necessary that the probability distributions under H_0 and H_1 be distinctly separated with as little overlap as possible. To achieve this it is necessary that $(ETM/2)\sqrt{M}N_0T/2 = \sqrt{M}E/N_0$ be large.

In order to determine the range of M for which (6.71) is a useful approximation, plots of P_d vs E/N_0 are presented in Figure 6.10 for different

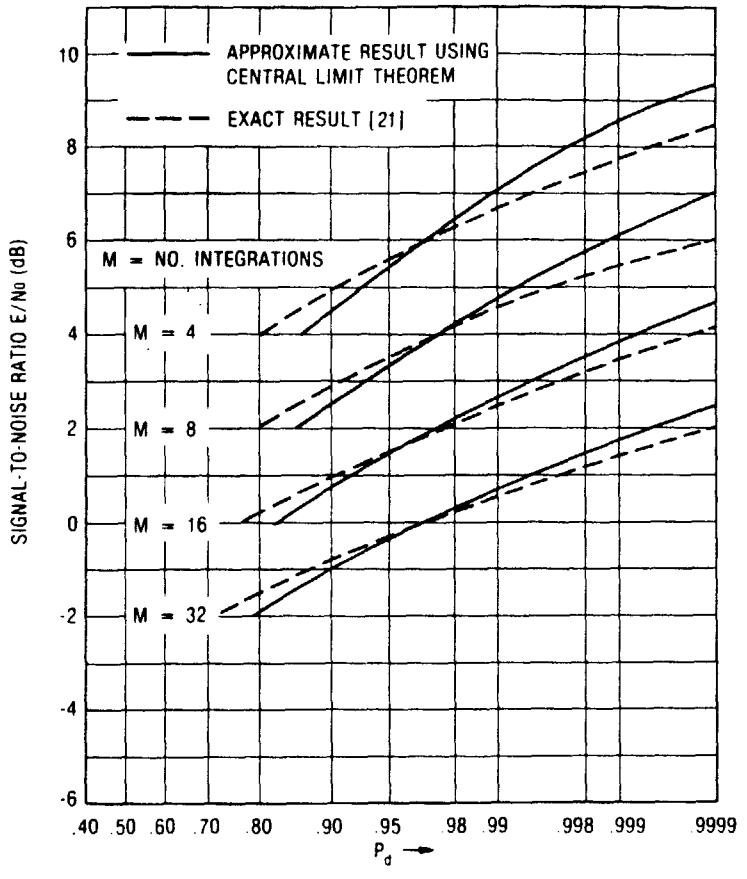


Figure 6.10 Comparison of SNR requirements with noncoherent integration ($P_{fa} = 10^{-2}$).

values of $M \geq 4$ using (6.71) along with more exact results from [21] which are based on the $2M$-degree-of-freedom noncentral chi-square distribution. The comparison in Figure 6.10 is for $P_{fa} = 10^{-2}$. In all cases the value of E/N_0 required to obtain a given value of P_d as determined using (6.71) is within 1 dB of the value obtained using the more exact solution.

6.4.2 Coherent Integration

In this case an exact solution is obtained. For predetection integration the output of the square-law detector is (6.53).

$$\left(\frac{MAT}{2}\cos\theta + \sum X_i\right)^2 + \left(\frac{MAT\sin\theta}{2} + \sum_{i=1}^{M} Y_i\right)^2 \tag{6.72}$$

which has a noncentral chi-square distribution with two degrees of freedom. Now in general, a noncentral chi-square random variable z with N degrees of freedom is of the form

$$z = \sum_{i=1}^{N} (B_i + x_i)^2 \tag{6.73}$$

where the B_i are constants, the x_i are independent, mean-zero Gaussian, with $\mathrm{Var}(x_i) = \sigma^2$, and z has a probability density [21]

$$P(z) = \frac{1}{2\sigma^2}\left(\frac{z}{\beta}\right)^{(N-2)/4} \exp\left[-\frac{(z+\beta)}{2\sigma^2}\right] I_{N/2-1}\left[\frac{(z\beta)^{1/2}}{\sigma^2}\right] \tag{6.74}$$

where $\beta = \Sigma_{i=1}^{N} B_i^2$, and $I_{N/2-1}$ is the modified Bessel function of order $N/2-1$. In the case of interest here this takes a very simple form because $N = 2$. Thus, $B_1 = \frac{1}{2}MAT\cos\theta$, $B_2 = \frac{1}{2}MAT\sin\theta$, $x_1 = \Sigma_{i=1}^{M} X_i$, $x_2 = \Sigma_{i=1}^{M} Y_i$, and x_1 and x_2 are each independent, mean-zero Gaussian with variance $\sigma^2 = MN_0T/4$. It can be shown using $I_0(0) = 1$ (see Exercise 6.9) that

$$P_{fa} = \int_{\eta^2}^{\infty} \exp\left(-\frac{z}{2\sigma^2}\right)\frac{dz}{2\sigma^2} = \exp\left(-\frac{\eta^2}{2\sigma^2}\right) \tag{6.75}$$

where for the square-law detector the threshold has been defined as η^2, and

$$\begin{aligned} P_d &= \int_{\eta^2}^{\infty} \exp\left(-\frac{z+\beta}{2\sigma^2}\right) I_0\left[\frac{(z\beta)^{1/2}}{\sigma^2}\right]\frac{dz}{2\sigma^2} \\ &= \int_{(-2\ln P_{fa})^{1/2}}^{\infty} x\exp\left(-\frac{x^2+\alpha^2}{2}\right) I_0(x\alpha)\,dx \end{aligned} \tag{6.76}$$

where $\alpha^2 = 2ME/N_0$. Equations (6.75) and (6.76), in which no approximations have been made, are seen to be identical with (6.19) and (6.21) with $2E/N_0$ in the latter case being replaced by $2ME/N_0$. Thus there is an effective improvement in SNR by a factor of M which goes over directly into the calculation of P_d, and the curves of Figure 6.6 apply exactly with ME/N_0 substituted for E/N_0 on the vertical axis. This is the expected result. Coherent predetection integration increases SNR by a factor of M and, aside from this change, the subsequent nonlinear operation reproduces exactly the results of Section 6.2.

For generalized matched filtering, by (6.22), (6.23), (6.63) and (5.52)

$$P_d = \frac{1}{2}\left[1 + \mathrm{erf}\left(\sqrt{\frac{2ME}{N_0}} - \gamma\right)\right] \tag{6.77}$$

where, as before, γ is determined from P_{fa} using

$$P_{fa} = \frac{1}{2}\left[1 - \mathrm{erf}(\gamma)\right], \quad \eta = 2\gamma\sqrt{\frac{ME}{N_0}} \tag{6.78}$$

The latter two results, (6.75), (6.76) and (6.77), (6.78) illustrate that the difference between coherent-integration employing square-law detection and generalized matched filtering is identical to that between coherent and noncoherent detection of a single pulse as discussed in Section 6.2.

6.5 SUMMARY OF COHERENT AND NONCOHERENT DETECTION AND PROCESSING

Coherent detection of a single pulse requires employment of a coherent matched filter, for which the exact value of the phase of the input signal must be known. For noncoherent detection of a single pulse the filter bandwidth is matched to that of the signal but the phase is unknown. The difference in the two cases in terms of the transmitted pulse energy required to achieve a given value of P_d for a specified value of P_{fa} becomes significant ($a \sim 3$ dB advantage in the coherent case) only for small values of E/N_0—say in the vicinity of 3 dB. This occurs because in the coherent case the quadrature noise term is eliminated, whereas for noncoherent detection both the in-phase and quadrature noise terms come into play. However, for $E/N_0 > 10$ dB this becomes less important because the quadrature-channel noise under this condition has very little effect on the fluctuations of the output signal-plus-noise envelope (or envelope squared), and is therefore effectively eliminated. For such values of E/N_0 the difference in SNR required to achieve a given value of P_d for a fixed P_{fa} between coherent and noncoherent detection is about 1 dB, and eventually becomes negligible as E/N_0 increases further.

For a fixed value of single-pulse energy E the detection capability of a sensor can be increased by employing a waveform consisting of a train of M pulses and adding, or integrating, their cumulative responses. If integration is done after detection—postdetection or noncoherent integration—which destroys the phase relationship between successive pulses, the signal pulses add incoherently and the integrated SNR is increased by a factor of $\sqrt{M}$. However, if the phase relationship between pulses is maintained by employing stable local oscillators, and the integration is implemented prior to detection, the SNR is improved by a factor of M, which of course can amount to a very large difference (see Exercises 6.14, 6.15, 6.16). In this it is not necessary that the absolute value of the signal phase be known. The phase however could be measured and, ideally, with the use of this information a filter matched to the entire M-pulse waveform could be implemented. The relative advantage thereby obtained however is identical to that between coherent and noncoherent detection of a single pulse, of the order of 3 dB in transmitted signal energy for $ME/N_0 \leq 2$, and essentially negligible for $ME/N_0 > 10$. As a result, exact phase knowledge is not important in coherent integration, because integration is generally employed expressly for the purpose of achieving large values of ME/N_0. It should be noted however that there might be reasons other than signal detection for

TABLE 6.1

Single Pulse

COHERENT DETECTION

$$P_{fa} = \frac{1}{2}[1 - \mathrm{erf}(\gamma)]$$

$$P_d = \frac{1}{2}\left[1 + \mathrm{erf}\left(\sqrt{\frac{E}{N_0}} - \gamma\right)\right]$$

NONCOHERENT DETECTION

$$P_d = \int_{(-2\ln P_{fa})^{1/2}}^{\infty} \exp\left(-\frac{x^2+\alpha^2}{2}\right) I_0(x\alpha)\,dx$$
$$= Q(\alpha, -2\ln P_{fa})$$

where $\alpha = 2E/N_0$ and $Q(\alpha, \beta)$ is Marcum's Q function

M-Pulse Waveform

NONCOHERENT INTEGRATION: APPROXIMATE SOLUTION FOR $M \geq 4$

$$P_{fa} = \frac{1}{2}[1 - \mathrm{erf}(\gamma)]$$

$$P_d = \frac{1}{2}\left\{1 + \mathrm{erf}\left[\frac{\sqrt{\frac{M}{2}}\,\frac{E}{N_0} - \gamma}{\left(1 + \frac{2E}{N_0}\right)^{1/2}}\right]\right\}$$

COHERENT INTEGRATION

$$P_d = \int_{(-2\ln P_{fa})^{1/2}}^{\infty} \exp\left(-\frac{x^2+\alpha^2}{2}\right) I_0(\alpha, x)\,dx$$
$$= Q(\alpha_M, -2\ln P_{fa})$$

where $\alpha_M = ME/N_0$

Generalized Matched Filtering for M Pulses

$$P_{fa} = \tfrac{1}{2}[1 - \mathrm{erf}(\gamma)]$$

$$P_d = \frac{1}{2}\left[1 + \mathrm{erf}\left(\sqrt{\frac{ME}{N_0}} - \gamma\right)\right]$$

which knowledge of the signal phase might be of interest. This is discussed in Chapter 7.

A summary of the quantitative performance results of this chapter are presented in Table 6.1. Although a specific form of the signal was used, the results are perfectly general. This is demonstrated in Chapter 10, where the same results are obtained using a generalized complex signal. We note that

calculations of P_d vs P_{fa} vs E/N_0 and M can in all cases be carried out by using Figure 6.6 and the table of values of the error function in the Appendix. For predetection integration, the vertical axis of Figure 6.6 should be interpreted as ME/N_0 rather than E/N_0.

EXERCISES FOR CHAPTER 6

6.1 A real bandpass function $x(t)$ can be written as $R[h(t)e^{i2\pi f_0 t}]$ where $h(t)$ can be complex. If $y(t) = \mathrm{Re}[g(t)e^{i2\pi f_0 t}]$ show that

$$\int_{-\infty}^{\infty} x(t)y(t)\,dt = \frac{1}{2B}\sum_{n=-\infty}^{\infty}\left[h_R\left(\frac{n}{B}\right)g_R\left(\frac{n}{B}\right) + h_I\left(\frac{n}{B}\right)g_I\left(\frac{n}{B}\right)\right]$$

by making use of the carrier sampling theorem (4.15). It is assumed that $x(t)$ and $y(t)$ are observed at the output of a bandpass filter such as shown in Figure 6.1. Recall that the Nyquist rate here is $\Delta t = 1/B$.

6.2 For input $A \sin 2\pi f_0 t + B \cos 2\pi f_0 t$ determine the outputs of detectors in Figure 6.4a,b,c. Show how low-pass filtering or video amplification produces the desired results. Show that b produces a single harmonic at $2f_0$ whereas the others produce an infinite number of harmonics.

6.3 For $s(t) = P_T(t)\cos(2\pi f_0 t + \theta)$ show that the response of a matched filter appears as in Figure 6.5a, whereas for a filter matched in amplitude only the output appears as in Figure 6.5b (assume $\theta \neq 0$).

6.4 For input $A\cos(2\pi f_0 t + \theta)$ calculate the SNR as defined in (5.61) at the output of
a. a filter $h(t) = P_T(t)\cos[2\pi f_0(T - t) + \theta]$
b. a filter $h(t) = P_T(t)\cos 2\pi f_0(T - t)$
Assume noise to be white, Gaussian with PSD $N_0/2$.

6.5 The Rice distribution in (6.17) differs from that in (3.53) in that in (3.53) the variable r has the dimensions of volts whereas in (6.17) the dimension of z are volts $\times\, T/2$ which is the effect of the filter. Find the appropriate transformation of the variable r so that (3.53) reduces to (6.17) and demonstrate this.

6.6 Show that $E(X_i^4) = E(Y_i^4) = 3N_0^2T^2/16$ where X_i and Y_i are given by (6.37). Using (6.40), the result of (6.4) and $E[n(t_1)n(t_2)] = \frac{1}{2}N_0\delta(t_2 - t_1)$, also show that X_i^2 and Y_i^2 are uncorrelated; that is $E(X_i^2Y_i^2) = E(X_i^2)E(Y_i^2)$.

6.7 Verify that the ith output of generalized matched filter is $\frac{1}{2}AT + X_i$; that is, the quadrature noise term vanishes.

6.8 Evaluate (6.67) and show that (6.66) reduces to (6.68).

6.9 Show that chi-square with two degrees of freedom reduces to the Rice distribution.

6.10 An observation takes place over 10 s. For $P_{fa} = 10^{-6}$ and PRF = 150 pulses/s write down an expression for the probability of k false alarms during this interval. What would be the expected value of k, and the mean time between false alarms. Repeat for $P_{fa} = 10^{-4}$ (see Section 11.4).

6.11 The dimensions of the threshold η for a matched filter and for ideal noncoherent detection are different. For $\frac{1}{2}N_0 = 10^{-6}$ W/Hz find the value of E for a coherent matched filter such that $P_{fa} = 10^{-4}$ can be obtained with a threshold of $\eta = 22$. What would be the resulting value of P_d?

For the same value of P_d and P_{fa} find the required value of E/N_0 for an envelope detector. Assume the noise is limited in bandwidth to the signal bandwidth. Find the required value of received signal power for a threshold of $\eta = 22$. Discuss the difference in units.

6.12 A target is to be observed for 10 s with a sensor operating with a PRF of 100. It is desired that the probability of more than one false alarm during this interval be $\leq 10^{-4}$. The detection probability is to be 0.998. Define the necessary system parameters (i.e. SNR and threshold) to achieve this for a coherent system (see Section 11.4).

6.13 For $E/N_0 = 20$ find P_d for $P_{fa} = 10^{-4}$ and 10^{-6} for single pulse coherent and noncoherent detection. Repeat for $E/N_0 = 30$. You will need tables of the Q function for this.

6.14 A system employs postdetection integration. What values of M are required to achieve a detection probability of $P_d = 0.99$ for $P_{fa} = 10^{-3}$ for values of E/N_0 of 0.25, 2, 10 and 100.

6.15 Repeat Exercise 6.13 assuming predetection integration and compare results.

6.16 Repeat Exercise 6.14 for generalized matched filtering and compare results.