A. Algorithmic Details

The purpose of this appendix is to briefly describe the algorithmic underpinnings of SDMA. An extensive technical literature concerning direction of arrival (DOA) estimation exists, we refer the reader to the bibliography contained in the Phase I proposal for a starting point. Here, the emphasis is on the ***ESPRIT*** (Estimation of Signal Parameters via Rotational Invariance Techniques) algorithm whose accuracy and computational simplicity make it particularly well suited to SDMA.

The Narrowband Data Model

Though ***ESPRIT*** is generally applicable to a wide variety of problems, for obvious reasons the discussions herein focus on DOA estimation. Data from an array of sensors are collected and the objective is to locate point sources assumed to be radiating energy that is detectable by the sensors. Herein, the transmission medium is assumed to be isotropic and non-dispersive so that the radiation propagates in *straight lines*, and the sources are assumed to be in the *far-field* of the array. For the ranges of RF's considered herein, this assumption is easily shown to valid. Consequently, the radiation impinging on the array is in the form of a sum of *plane waves*. For simplicity, it will initially be assumed that the problem is planar, thus reducing the location parameter space to a single-dimensional subset of $\Re$, *i.e.*, $\theta_i \in [-\pi, \pi]$, where θ_i is the direction-of-arrival (DOA) of the ith source. Assuming the signals are *narrowband* and have the same *known* center frequency, ω_0, $x_k(t)$, the complex signal output of the kth sensor at time t, can be written as

$$x_k(t) = \sum_{i=1}^{d} a_k(\theta_i) s_i(t - \tau_{ki}) = \sum_{i=1}^{d} a_k(\theta_i) s_i(t) e^{-j\omega_0 \tau_{ki}} , \tag{A.1}$$

where τ_{ki} is the propagation delay between a reference point and the kth sensor for the ith wavefront impinging on the array from direction θ_i, $a_k(\theta_i)$ is the corresponding sensor element complex response (gain and phase) at frequency ω_0, and there are assumed to be d point sources present.

Employing vector notation for the outputs of the m sensors, the *data model* becomes

$$\mathbf{x}(t) = \sum_{i=1}^{d} \mathbf{a}(\theta_i) s_i(t) , \tag{A.2}$$

where $\mathbf{a}(\theta_i) = [a_1(\theta_i)e^{-j\omega_0\tau_1(\theta_i)}, \ldots, a_m(\theta_i)e^{-j\omega_0\tau_m(\theta_i)}]^T$, often termed the *array steering vector* for direction θ_i. Setting $\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_d)]$, $\mathbf{s}(t) = [d_1(t), \ldots, d_d(t)]^T$, and adding measurement noise $\mathbf{n}(t)$, the *measurement model* for the passive sensor array narrowband signal processing problem is

$$\mathbf{z}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t) + \mathbf{n}(t) . \tag{A.3}$$

Note that $\mathbf{z}(t), \mathbf{n}(t) \in \mathbb{C}^m$, $\mathbf{s}(t) \in \mathbb{C}^d$, and $\mathbf{A}(\boldsymbol{\theta}) \in \mathbb{C}^{m \times d}$, and it is assumed that $m > d$.

ESPRIT enjoys a significant reduction in computational complexity relative to most DOA algorithms, obtained by imposing a simple constraint on the structure of the sensor array. The constraint is most easily described by an example. Consider a planar array of arbitrary geometry composed of $M/2$ sensor *doublets* as shown in Figure A-1. The

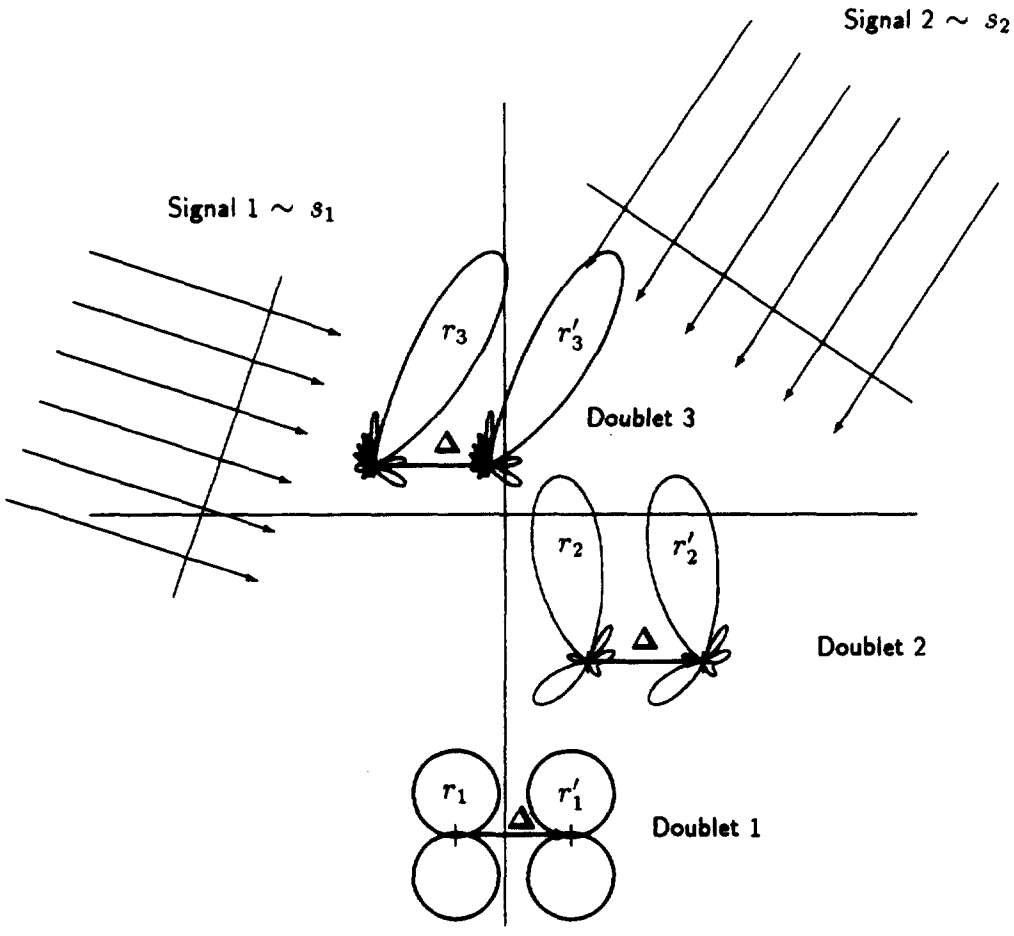

Figure A-1: Sensor Array Geometry for Multiple Source DOA Estimation Using ***ESPRIT***

elements in each doublet have identical sensitivity patterns and are translationally separated by a known constant displacement vector $\boldsymbol{\Delta}$. Other than the obvious requirement that each sensor have non-zero sensitivity in all directions of interest, the gain, phase, and polarization sensitivity of the elements in the doublet are arbitrary. Furthermore, there is no requirement that any of the doublets possess the same sensitivity patterns, though as discussed in [1], there are advantages to employing arrays with such characteristics such as uniform linear arrays (ULA's).

ESPRIT Data Model

Assume that there are $d \leq m$ (m is the number of sensors in each subarray) narrow-band sources[1] centered at frequency ω_0, and that the sources are located sufficiently far from the array such that in homogeneous isotropic transmission media, the wavefronts impinging on the array are planar. As before, the sources may be assumed to be stationary zero-mean random processes or deterministic signals. Additive noise is present at all $M = 2m$ sensors and as before is assumed to be a stationary zero-mean random process with a *spatial* covariance $\sigma^2\mathbf{I}$. There is no loss of generality in comparison with the assumption that the covariance is $\sigma^2\mathbf{\Sigma}_n$. As long as $\mathbf{\Sigma}_n$ is known, *pre-whitening* of the measurement noise can be performed, or generalized singular value or eigendecompositions can be employed.

To describe mathematically the effect of the *translational invariance* of the sensor array, it is convenient to describe the array as being comprised of two subarrays, Z_0 and Z_1, identical in every respect although physically displaced (not rotated) from each other by a known displacement vector $\mathbf{\Delta}$. The signals received at the ith doublet can then be expressed as

$$\begin{aligned} z_i^{(0)}(t) &= \sum_{k=1}^{d} s_k(t)a_i(\theta_k) + n_{0i}(t), \\ z_i^{(1)}(t) &= \sum_{k=1}^{d} s_k(t)e^{j\omega_0\Delta\sin\theta_k/c}a_i(\theta_k) + n_{1i}(t), \end{aligned} \tag{A.4}$$

where θ_k is the direction of arrival of the kth source relative to the direction of the translational displacement vector $\mathbf{\Delta}$.

Combining the outputs of each of the sensors in the two subarrays, the received data vectors can be written as follows:

$$\mathbf{z}_0(t) = \mathbf{A}_0\mathbf{s}(t) + \mathbf{n}_0(t)\,, \tag{A.5}$$

$$\mathbf{z}_1(t) = \mathbf{A}_0\mathbf{\Phi}\mathbf{s}(t) + \mathbf{n}_1(t)\,, \tag{A.6}$$

where the vector $\mathbf{s}(t)$ is the $d \times 1$ vector of impinging signals (wavefronts) as observed at the reference sensor of subarray Z_0, and $\mathbf{A}_0$ is the array response vector associated with Z_0. The individual signals can be temporally correlated, *i.e.*, $E\{s_i(t)s_j^*(t)\} \neq 0,\ i \neq j$. Coherent (or fully correlated) sources can easily be handled as discussed in the next section if there are multiple invariances in the array. However, coherent sources can not be uniquely located with only a single array invariance and unknown array manifold vectors. The matrix $\mathbf{\Phi}$ is a diagonal $d \times d$ matrix of the phase delays between the doublet sensors for the d wavefronts and may be written as $\mathbf{\Phi} = \mathrm{diag}[e^{j\gamma_1}, \ldots, e^{j\gamma_d}]$, where $\gamma_k = \omega_0\Delta\sin\theta_k/c$. $\mathbf{\Phi}$ is a unitarymatrix (operator) that relates the measurements from subarray Z_0 to those from subarray Z_1.

[1]MUSIC imposes the requirement $d < 2m$, and can therefore can handle roughly twice as many sources as ***ESPRIT*** in general. For uniform linear arrays, however, ***ESPRIT*** can handle as many sources as MUSIC by employing *overlapping* subarrays.

Defining the total array output vector as $\mathbf{z}(t)$, the subarray outputs can be combined to yield

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{z}_0(t) \\ \mathbf{z}_1(t) \end{bmatrix} = \bar{\mathbf{A}}\mathbf{s}(t) + \mathbf{n}_z(t), \tag{A.7}$$

where

$$\tag{A.8}$$

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_0 \\ \mathbf{A}_0\mathbf{\Phi} \end{bmatrix}, \quad \mathbf{n}_z(t) = \begin{bmatrix} \mathbf{n}_0(t) \\ \mathbf{n}_1(t) \end{bmatrix}. \tag{A.9}$$

It is the structure of $\bar{\mathbf{A}}$ that is exploited to obtain estimates of the diagonal elements of $\mathbf{\Phi}$ *without prior knowledge of* $\mathbf{A}_0$.

In its single invariance, eigenvalue decomposition version, the ***ESPRIT*** algorithm operates as follows. Details can be found in [1, 2].

1. Obtain an estimate of $\mathbf{R}_{zz}$, denoted $\widehat{\mathbf{R}}_{zz}$, from the measurements.

2. Compute the eigen-decomposition of $\widehat{\mathbf{R}}_{zz}$

$$\widehat{\mathbf{R}}_{zz}\bar{\mathbf{E}} = \bar{\mathbf{E}}\mathbf{\Lambda},$$

   where $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_M\}$, $\lambda_1 \geq \cdots \geq \lambda_M$, and $\bar{\mathbf{E}} = [\mathbf{e}_1 \,\vdots\, \cdots \,\vdots\, \mathbf{e}_M]$.

3. Estimate the number of sources $\hat{d}$ using the minimum description length statistical criterion.

4. Obtain the signal subspace estimate $\hat{\mathcal{S}}_z = \mathcal{R}\{\widehat{\mathbf{E}}_s\} \stackrel{\text{def}}{=} [\mathbf{e}_1 \,\vdots\, \cdots \,\vdots\, \mathbf{e}_{\hat{d}}]$ and decompose it using a *selection matrix* $\mathbf{J}$ according to a particular choice of subarrays,

$$\mathbf{J}\widehat{\mathbf{E}}_s = \begin{bmatrix} \widehat{\mathbf{E}}_0 \\ \widehat{\mathbf{E}}_1 \end{bmatrix}.$$

5. Compute the eigendecomposition ($\lambda_1 > \ldots > \lambda_{2\hat{d}}$),

$$\widehat{\mathbf{E}}_{01}^*\widehat{\mathbf{E}}_{01} \stackrel{\text{def}}{=} \begin{bmatrix} \widehat{\mathbf{E}}_0^* \\ \widehat{\mathbf{E}}_1^* \end{bmatrix} [\,\widehat{\mathbf{E}}_0 \,\vdots\, \widehat{\mathbf{E}}_1\,] = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^*,$$

   and partition $\mathbf{V}$ into $\hat{d} \times \hat{d}$ submatrices,

$$\mathbf{V} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix}.$$

6. Calculate the eigenvalues of $\mathbf{\Psi} = -\mathbf{V}_{12}\mathbf{V}_{22}^{-1}$,

$$\hat{\phi}_k = \lambda_k(-\mathbf{V}_{12}\mathbf{V}_{22}^{-1}), \quad \forall\, k = 1, \ldots, \hat{d}.$$

7. Estimate $\hat{\theta}_k = f^{-1}(\hat{\phi}_k)$; *e.g.*, for DOA estimation, $\hat{\theta}_k = \sin^{-1}\{c\arg(\hat{\phi}_k)/(\omega_0\Delta)\}$.

Signal Copy

In signal waveform estimation (often termed ***signal copy***), one signal of interest (SOI) is often assumed to be present along with a number of other directional sources (*jammers* or *interferers*) and additive noise. The total number of sources and their DOA's are assumed to be known or to have been estimated (*e.g.* via ***ESPRIT***). As stated earlier, the received signal $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta}_0)\mathbf{s}(t) + \mathbf{n}(t), \tag{A.10}$$

$$\mathbf{X}_N = [\mathbf{x}(1), \ldots, \mathbf{x}(N)] = \mathbf{A}(\boldsymbol{\theta}_0)\mathbf{S}_N + \mathbf{N}_N, \tag{A.11}$$

where $\mathbf{A}(\boldsymbol{\theta}_0)$ is the collection of sensor responses parameterized by $\boldsymbol{\theta}$. The signals $\mathbf{s}(t)$ are assumed to be independent of the additive noise $\mathbf{n}(t)$.

The deterministic signal model [3] assumes that the signal waveforms, $\mathbf{S}_N$, are unknown deterministic quantities to be estimated. The resulting normalized negative log-likelihood function has the form

$$\begin{aligned} V_N &= m\log(\pi\sigma^2) \\ &+ \frac{\sigma^{-2}}{N}\mathrm{Tr}\{(\mathbf{X}_N - \mathbf{A}(\boldsymbol{\theta})\mathbf{S}_N)^*(\mathbf{X}_N - \mathbf{A}(\boldsymbol{\theta})\mathbf{S}_N)\}. \end{aligned} \tag{A.12}$$

Minimizing (A.12) with respect to $\boldsymbol{\theta}$ and $\mathbf{S}_N$ is equivalent to minimizing

$$\frac{1}{N}\|\mathbf{X}_N - \mathbf{A}(\boldsymbol{\theta})\mathbf{S}_N\|_F^2. \tag{A.13}$$

This is a separable problem in $\boldsymbol{\theta}$ and $\mathbf{S}_N$, and the estimate of $\mathbf{S}_N$ is given by the least squares solution of (A.13),

$$\begin{aligned} \hat{\mathbf{S}}_N &= (\mathbf{A}^*(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta}))^{-1}\mathbf{A}^*(\boldsymbol{\theta})\mathbf{X}_N = \mathbf{A}^\dagger(\boldsymbol{\theta})\mathbf{X}_N \\ &= \mathbf{W}_{\mathrm{DET}}^*\mathbf{X}_N\,. \end{aligned} \tag{A.14}$$

Substituting this back in (A.13) leads to

$$\min_{\boldsymbol{\theta}} \mathrm{Tr}\{(\mathbf{I} - \mathbf{A}(\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*)\widehat{\mathbf{R}}_{xx}\} = \min_{\boldsymbol{\theta}} \mathrm{Tr}\{\mathbf{P}_{\mathbf{A}(\boldsymbol{\theta})}^{\perp}\widehat{\mathbf{R}}_{xx}\}, \tag{A.15}$$

where $\widehat{\mathbf{R}}_{xx}$ is the sample covariance $\mathbf{X}_N\mathbf{X}_N^*/N$. It is interesting to note that the signal parameter estimates $\widehat{\boldsymbol{\theta}}$ obtained from the deterministic ML method are not *efficient*, *i.e.*, the method does not attain the deterministic CRB [4]. This is a consequence of the fact that the estimates of the signal waveforms themselves are inconsistent.

Bibliography

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[4] P. Stoica and A. Nehorai, MUSIC, maximum likelihood and Cramér-Rao bound, *IEEE Trans. ASSP*, **ASSP-37**:720–741, May 1989.

Exhibit H

ALL-STATE LEGAL SUPPLY CO. 1-800-222-0510 EDR11 RECYCLED

EXHIBIT H

		Conventional		SDMA	
Urbanization	Total Area	Cell Radius	# of Cells	Cell Radius	# of Cells
City	3080 km^2	0.5 *km*	3922	0.7 *km*	2001
Urban	57750 km^2	1.5 *km*	8170	3.0 *km*	2042
Rural	3224170 km^2	4.0 *km*	5482	10.0 *km*	877
Totals	3285000 km^2		17574		4920
TOTAL COST			$2900M		$1900M

Table 1-1: Cost Advantage of SDMA over Conventional DCS-1800 Systems