H. Provisions are Necessary to Control the Power Output of Very Short Duration "Impulse" Emissions

Taken together, the modifications and clarifications to the definitions, limits, and measurement procedures proposed above provide an accurate and practical mechanism for controlling the interference potential of high-speed digital transmissions which transmit pulses of significant duration. However, under the current Rules, there is also the potential for devices that transmit isolated "impulse" signals that are very short in duration (*e.g.,* 5-10 ns). WINForum has held several discussions with NTIA regarding appropriate measures to regulate the interference potential of U-NII devices, and both agree that the impact of impulse transmissions (*e.g.*, for pulse-position modulation) needs to be considered.

If an extremely wideband (*e.g.,* 100 MHz bandwidth) impulse of short duration (*e.g.*, 10 ns) is applied to a 1 MHz measurement filter, the filter output is essentially the "impulse response" of the filter, which will have a duration inversely proportional to the filter bandwidth and an amplitude proportional to the impulse amplitude. The energy in the output signal will be proportional to the power of the impulse multiplied by the filter bandwidth, so the average power of the filter output will be proportional to the impulse power multiplied by the square of the filter bandwidth (because the response duration varies inversely with the filter bandwidth). However, the power of the impulse cannot be determined because the impulse duration remains unknown. An instrument with a maximum resolution bandwidth on the order of 2 to 3 MHz is therefore not adequate to determine the power of the impulse signal, and must be supplemented with a wideband instrument capable of following the impulse envelope. Measurement of power spectral density is also problematic, because the peak power spectral density (and the frequencies at which it occurs) will depend on the impulse repetition rate.

WINForum concludes that special considerations for measurement techniques are needed to accommodate impulse transmissions. WINForum therefore plans to continue to discuss this issue with NTIA, with a goal of jointly developing suitable test procedures for measuring the total power output and power spectral density of impulse transmissions, which can be recommended to the Commission. WINForum believes that it will be necessary to specify in the Rules that the manufacturer is required to disclose, in its application for Part 15 Certification, details such as the modulation format, minimum pulse duration, minimum and maximum pulse repetition rate, spectral characteristics under expected conditions of operation, *etc*, so that the Commission can apply the appropriate test procedures.

I. The Definition of U-NII Devices Should be Clarified to Require Digital Modulation

Section 15.403(a) defines U-NII devices as "Intentional radiators operating in the frequency bands 5.15-5.35 GHz and 5.725-5.825 GHz that provide a wide array of wideband, high data rate, digital, mobile and fixed communications for individuals, businesses, and institutions." WINForum believes that the Commission's intent is to foster the development of advanced wideband digital radio technologies for applications requiring high data rate wireless connectivity. Therefore, to minimize unintended use of this spectrum, WINForum proposes that the definition of U-NII devices be modified as follows to specifically state that the use of this spectrum is restricted to systems using digital modulation techniques.

> *(a) U-NII devices [Unlicensed].* Intentional radiators operating in the frequency bands 5.15-5.35 GHz and 5.725-5.825 GHz that use wideband digital modulation techniques and provide a wide array of high data rate mobile and fixed communications for individuals, businesses, and institutions.

III. CONCLUSION

WINForum believes that the technical clarifications and modifications suggested herein will serve the public interest. Specifically, the proposals herein further the Commission's intent by increasing the flexibility of manufacturers to design functional and innovative equipment, by ensuring that the minimal regulatory framework set forth by the Commission is fully realized, and finally by not increasing any potential for harmful interference to other spectrum users. WINForum once again commends the Commission for its expeditious action on this item, and urges the Commission adopt the recommendations set forth herein.

Respectfully submitted,

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Attachment A

WIDEBAND EMISSIONS THROUGH A NARROWBAND FILTER AND THE IMPLICATIONS ON MEASUREMENT OF POWER SPECTRAL DENSITY USING A SPECTRUM ANALYZER

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EXECUTIVE SUMMARY

For interference-control purposes, the relevant measure of a signal's "spectrum" is the power spectral density (PSD), in watts/Hz. PSD is normally measured with a spectrum analyzer, which effectively sweeps a filter across the signal bandwidth. The power output of the filter controls the trace on the analyzer's display. To accurately determine the shape of the signal spectrum, the bandwidth of this filter, which is the "resolution bandwidth" (RBW) must be small compared to the signal bandwidth.

The signal power captured by the resolution filter at any instant in time, when it is centered on a particular frequency, represents the interference power which would be seen by a receiver with the same bandwidth and center frequency as the resolution filter. However, if the bandwidth of the resolution filter is much narrower than the signal bandwidth, the filter power output fluctuates in a manner very similar to that of Gaussian noise. This is easily observed on a spectrum analyzer, and it is shown here that as the filter bandwidth becomes increasingly narrow, the statistical behavior (i.e., the probability distribution) of the filter power output does in fact approach the Gaussian case (Rayleigh-distributed envelope voltage and exponentially-distributed envelope power). This is the reason that a single-sweep trace of a signal's spectrum appears "rough" on the spectrum analyzer. It is also the reason that a wideband signal, such as a direct-sequence spread spectrum signal, appears to be Gaussian noise to a narrowband receiver.

This paper provides a mathematical analysis which (1) shows that the average filter power output is the actual PSD at the filter center frequency multiplied by the filter noise bandwidth and (2) demonstrates mathematically the reason that the output of a narrow filter behaves like Gaussian noise. It also describes the results of an experiment using a spectrum analyzer, a digital signal generator, and a noise generator which support the results of the analysis.

This behavior has significant implications for PSD measurement procedures, especially for the U-NII (Unlicensed National Information Infrastructure) devices recently authorized by the FCC in ET Docket 96-102. While in-band PSD limits are normally imposed on unlicensed devices (as in sections 15.247 and 15.319-15.232 of the FCC Rules), those limits do not tend to be the controlling factor for power output. For the U-

NII devices, the output power is controlled not only with a cap on total transmitted power, but also with a cap on the total power per MHz. Thus, for U-NII devices, PSD is effectively a primary controller of output power. Compliance with the 1-MHz PSD limit presumably would be verified by measuring the power output within a 1-MHz bandwidth. Since U-NII devices are expected to have emission bandwidths on the order of 20 to 25 MHz, the power output of a measurement filter with a 1 MHz bandwidth will exhibit the noise-like variation described above. The peak-to-average ratio for the filter output can be on the order of 6 to 10 dB, depending on the randomness of the modulating signal. Thus, if the PSD were to be measured on an analyzer by using a 1-MHz RBW with a "peak hold" or "max hold" trace mode (commonly used to generate a "smooth" spectral trace on the analyzer), the measured result will be 6 to 10 dB above the true PSD. If this measured maximum is required to comply with the 1-MHz power limit (e.g., 2.5 mW/MHz for the 5150-5250 MHz band), the effect will be to reduce the actual allowed PSD by 6 to 10 dB. For example, with an 8-dB reduction, the actual PSD would be about 0.4 mW/MHz. The total power output would be reduced proportionally, seriously compromising the ability to design useful systems. For example, the maximum power level in the 5150-5250 MHz band would become about 7.9 mW. To avoid this problem, it is necessary to accurately measure the true PSD, which requires averaging of the resolution filter power output.

From an interference-control perspective, the average filter power output (while the device is transmitting at its maximum level) is the appropriate quantity to control. The performance of a receiver typically is specified in terms of the bit error rate vs. E_b/N_0, where E_b is the average energy per bit and N_0 is the average power spectral density of Gaussian noise. Since the interference from the wideband device will appear to a narrowband receiver to be Gaussian noise, it is the actual PSD of the U-NII device (the effective N_0) which will determine the effect of the U-NII device on the "victim" device. If the interference into victim device is somewhat less variable than Gaussian noise, then the effect on the victim receiver will be correspondingly less severe than that of Gaussian noise with the same average power.

POWER SPECTRAL DENSITY

Definition of PSD

Consider a signal $s(t)$ that is a sample function of a random process $\mathbf{s}(t)$.[1] The autocorrelation function of $\mathbf{s}(t)$ is defined as:

$$R_{ss}(\tau) = E\left[\mathbf{s}(t)\mathbf{s}(t+\tau)\right] \tag{1}$$

where $E[\cdot]$ denotes expectation (the statistical average). The power spectral density (PSD) of $\mathbf{s}(t)$ is defined as the Fourier transform of the autocorrelation:

[1]The random (or stochastic) process $\mathbf{s}(t)$ is assume to be at least wide-sense stationary and ergodic.

$$\Phi_{ss}(f) = \int_{-\infty}^{\infty} R_{ss}(\tau) e^{-j2\pi f\tau} d\tau, \tag{2}$$

where $j = \sqrt{-1}$.

Determining the PSD from Observed Signal Samples

The fact that the PSD is defined in terms of an expected value suggests that some sort of averaging must be performed to infer the PSD of the process $\mathbf{s}(t)$ from the observed signal $s(t)$. The problem is similar in concept to estimating the mean of a random variable by averaging a large number of samples. Indeed, since $\Phi_{ss}(f)$ can be written as

$$\Phi_{ss}(f) = E\left[\int_{-\infty}^{\infty} \mathbf{s}(t)\mathbf{s}(t+\tau) e^{-j2\pi f\tau} d\tau\right], \tag{3}$$

it might be reasonable to expect that the PSD could be estimated by averaging samples of:

$$\Psi_T(f,t) \equiv \int_{-T}^{T} s(t)s(t+\tau) e^{-j2\pi f\tau} d\tau, \tag{4}$$

where T is large compared to the correlation time of the process $\mathbf{s}(t)$. In that case, it seems plausible that:

$$\Phi_{ss}(f) = E\left[\Psi_T(f,t)\right], \tag{5}$$

where averaging (expectation) is performed over different time samples of $\Psi_T(f,t)$. An estimate of the PSD then could be obtained using:

$$\hat{\Phi}_{ss}(f) = \frac{1}{K}\sum_{k=1}^{K} \Psi_T(f,t_k). \tag{6}$$

The variance of the estimation error can be made arbitrarily small by making K sufficiently large.

The reasoning given above suggests that the PSD can be estimated with the appropriate processing of samples of $\mathbf{s}(t)$ and time-averaging. However, the purpose of this paper is not to discuss spectral estimation in general,[2] but rather to explore methods for accurate measurement of the PSD using a spectrum analyzer. The main point to be taken here is that whatever technique and instrumentation are used to estimate the PSD, some sort of time-averaging will be necessary, because of the inherent nature of the PSD. This will become clearer in the discussion below, which relates the PSD to the Fourier transform of

[2]For background on spectral estimation, see *Signal Analysis*, A. Papoulis, McGraw-Hill, 1977, chapter 12.

an observed signal $s(t)$, which is a sample function of the underlying stochastic process $\mathbf{s}(t)$.

Relationship Between PSD and the Fourier Transform of a Random Signal

Assume that an observed segment of $s(t)$ is of duration T. The observed spectral distribution of $s(t)$ is given by the Fourier transform:

$$S_T(f) = \int_0^T s(t)e^{-j2\pi ft}\,dt \quad \text{volts/Hz.} \tag{7}$$

The instantaneous signal power is $|s(t)|^2$ and the total energy in the signal segment is:

$$E_{sT} = \int_0^T |s(t)|^2\,dt = \int_{-\infty}^{\infty} |S_T(f)|^2\,df \quad \text{joules.} \tag{8}$$

Note that the signal power $|s(t)|^2$ gives the energy distribution over time, while $|S_T(f)|^2$ gives the energy distribution over frequency; that is:

$$|s(t)|^2 = \frac{dE_{sT}}{dt} \quad \text{joules/sec (watts)} \tag{9}$$

$$|S_T(f)|^2 = \frac{dE_{sT}}{df} \quad \text{joules/Hz} \quad . \tag{10}$$

The "instantaneous" power spectral density can be viewed as the rate at which energy is accumulating per Hz at a given frequency:

$$\Phi_{sT}(f) = \frac{\partial}{\partial T}\left[|S_T(f)|^2\right] \qquad \text{watts/Hz} \tag{11}$$

Since $s(t)$ is an information-bearing signal, $\Phi_{sT}(f)$ cannot be time-invariant. Since the information embedded in $s(t)$ is unknown, $\Phi_{sT}(f)$ is itself a random process (varying with T) as discussed above. It seems reasonable to anticipate that the actual PSD $\Phi_{ss}(f)$ is the mean of that process, which suggests that $\Phi_{ss}(f)$ can be found by somehow averaging samples of $\Phi_{sT}(f)$.

This is in fact the case. As shown in the Appendix, $E[\Phi_{sT}(f)] = \Phi_{ss}(f)$ if T is sufficiently large compared to $1/B_s$, where B_s is the bandwidth of the process $\mathbf{s}(t)$. As also explained in the Appendix, under that condition, the distribution of the random

variable $\Phi_{sT}(f)$ approaches an exponential distribution. Thus, in the limit as $TB_s \to \infty$, $\Phi_{sT}(f)$ can be represented as:

$$\Phi_{sT}(f) = v\Phi_{ss}(f), \tag{12}$$

where v is an exponentially-distributed random variable with a mean of 1. It is also shown in the Appendix that the power output of the spectrum analyzer resolution filter is also a random variable that approaches an exponential in the limit.

MEASUREMENT OF THE PSD WITH A SPECTRUM ANALYZER

Spectrum Analyzer Operation

In effect, a spectrum analyzer sweeps a filter with a selected bandwidth across the frequency band of interest.[3] The bandwidth of this filter is the "resolution bandwidth" (RBW), selected by the operator. The selected RBW usually refers to the 3- or 6-dB bandwidth of the actual filter. The filter output is typically log-amplified and envelope-detected. The resulting baseband signal (after appropriate video processing) determines the vertical deflection of the signal trace on the display. Thus, the level shown on the display represents the envelope of the resolution filter output. The display therefore effectively shows the signal power through the filter versus the filter center frequency, subject to whatever video filtering and subsequent video processing is selected.

For analysis, the resolution filter will be represented by the transfer function $H(f)$, which has a center frequency f_0 at a particular time. If $g(t)$ represents the signal out of the filter at time t, the vertical magnitude of the display trace is either $g(t)$, the power output $|g(t)|^2$, or its dB value, depending on the display mode selected. The horizontal position of the trace at time t corresponds to the center frequency of the filter at that time.

Filter Noise Bandwidth and Normalization

The effective noise bandwidth of a filter is defined as:

$$B_N = \frac{\int_0^\infty |H(f)|^2 df}{|H(f_0)|^2}. \tag{13}$$

This means that if the filter input is white noise (i.e., spectrally flat), the average power output of the filter is the same as with a rectangular filter with bandwidth B_N and magnitude $|H(f_0)|$ at the center frequency. If the two-sided PSD of the input noise is

[3]In actuality, this is typically accomplished by mixing the signal to be analyzed with a swept local oscillator and passing the mixer output through a filter with a fixed center frequency.

$N_0/2$, the average power output is $N_0 B_N \left|H(f_0)\right|^2$ (since the magnitude of the filter transfer function is symmetric about $f = 0$). For this analysis, it will be assumed that $\left|H(f_0)\right| = 1$, so that the average power output of the filter is equal to the PSD of the input times the filter noise bandwidth. This means that for resolution bandwidths much less than the signal bandwidth, the average power level shown on the display generally will vary linearly with the filter bandwidth.

Resolution Filter Output

As shown in the Appendix, if the filter bandwidth B_N is much less than the signal bandwidth B_s, the distribution of the filter power output $P_f(t) = \left|g(t)\right|^2$ approaches an exponential distribution and $P_f(t)$ can be represented as:

$$P_f(t) = 2B_N \Phi_{ss}(f) v(t), \tag{14}$$

where $v(t)$ is an exponential random variable with an average value of 1. The factor of 2 reflects the fact that a two-sided PSD has been used in the analysis for mathematical symmetry. The probability distribution function (pdf) of v is:

$$p(v) = e^{-v} \tag{15}$$

As also shown in the Appendix, the autocorrelation of v is determined by the resolution filter shape. If, for example, $B_N = 1\,\text{MHz}$, the correlation time of $v(t)$, and therefore of $P_f(t)$, is on the order of $1\,\mu s$.

Fig. 1 shows the exponential distribution $\Pr\{v < x\} = 1 - e^{-x}$ vs. $10 \log x$.

Note that while levels more than about 8 dB above the mean are very unlikely, levels 20 dB below the mean can be expected to occur with some regularity.

Clearly, if $P_f(t)$ is averaged over a time interval that is large compared to its correlation time, a good estimate of the actual PSD $\Phi_{ss}(f)$ can be obtained. The averaging must be done on a linear power samples, rather than on dB samples. For an exponential random variable, the dB error that will result from averaging dB samples can be computed by finding the average of $u = 10 \log v$:

$$E[u] = \int_0^\infty 10 \log v \cdot e^{-v} dv = \frac{10}{\ln 10} \int_0^\infty \ln v \cdot e^{-v} dv = -2.51\,\text{dB} \tag{16}$$

Thus, dB average will be 2.51 dB less than the dB value of the linear average.

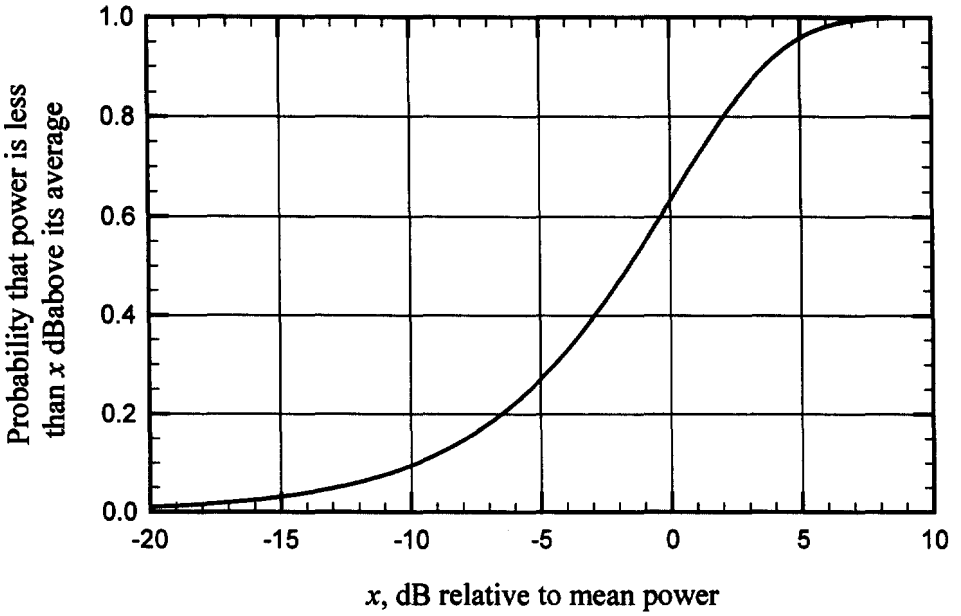

Figure 1: *The exponential distribution*

EXPERIMENTAL RESULTS

To test the validity of the theoretical results developed here, an experiment was conducted using an Anritsu MG3670B Digital Modulation Signal Generator and an HP 8651E Spectrum Analyzer. The signal uses $\pi/4$ DQPSK modulation (differential quadrature phase shift keying with a $\pi/4$ radian shift between successive symbols, to reduce envelope variation). The bit rate is 384 kb/s, so the symbol rate is 192 kbaud. The frame is 2.5 msec in duration and comprises 8 timeslots. For the measurements, the information field in each slot was filled with a pseudorandom sequence. In the tests described here, all slots were "on", so transmission was continuous except for inter-slot guard time. The RF power into the analyzer was –20 dBm (the generator was set to –19 dBm, and the loss in the cable and connectors was about 1 dB). This power level was verified by turning off the modulation and measuring the power of the CW signal at the analyzer.

Fig. 2 shows the signal spectrum, using a 10-kHz resolution bandwidth and a 1-MHz video bandwidth. Sample detection was used, and there was no video averaging or other processing, so Fig. 2 shows the "raw" output of the resolution filter on a single sweep. The "roughness" of the trace is a consequence of the variation of the filter power output, which is precisely the phenomenon described above. As can be seen, local variations in the trace on the order of 20 dB are common, and the variations sometimes reach 30 dB. The 10 kHz bandwidth was chosen because it analogous to using a 1-MHz resolution filter to measure a signal with a 20-MHz bandwidth.

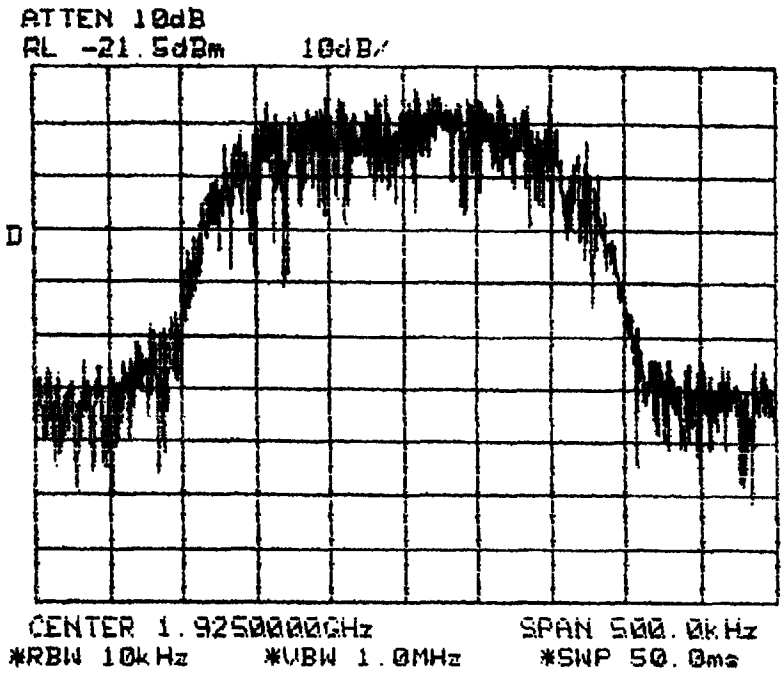

Figure 2: *Spectrum analyzer trace for 384 kb/s, π/4 DQPSK signal; single sweep, sample detection, RBW=10 kHz, VBW=1 MHz, total signal power is –20 dBm.*

Fig. 3 shows the trace for the same settings as in Fig. 2, except that the analyzer was in the zero-span mode with the resolution filter centered on the signal center frequency, and the scale was changed to 5 dB/division. Sample detection was also used in Fig. 3. This means that for each of the 600 "bins" in the horizontal sweep, a single sample of the filter power output was taken. The average power, computed using a built-in function of this particular analyzer, was –31.5 dBm. The upper edge of the grid was set at –21.5 dBm, so the average power is 2 vertical divisions below the top. For comparison, Fig. 4 shows the results of a simulation which generated 600 uncorrelated samples of an exponentially-distributed random variable. Note the similarity in appearance of Figs. 3 and 4 with respect to the apparent distribution of the power above and below the mean.

Fig. 5 shows a similar trace for a signal from a noise generator (the NoiseCom NC 6112). Again, the upper edge of the grid was set 10 dB above the average power. The similarity of Figs. 3, 4, and 5 suggests that the filter power output, even for a modulated digital signal, does indeed behave approximately as Gaussian noise (exponentially-distributed power). However, the power in Fig. 3 (from the digital signal) appears to be distributed over a *slightly* narrower range than the power from the noise generator.

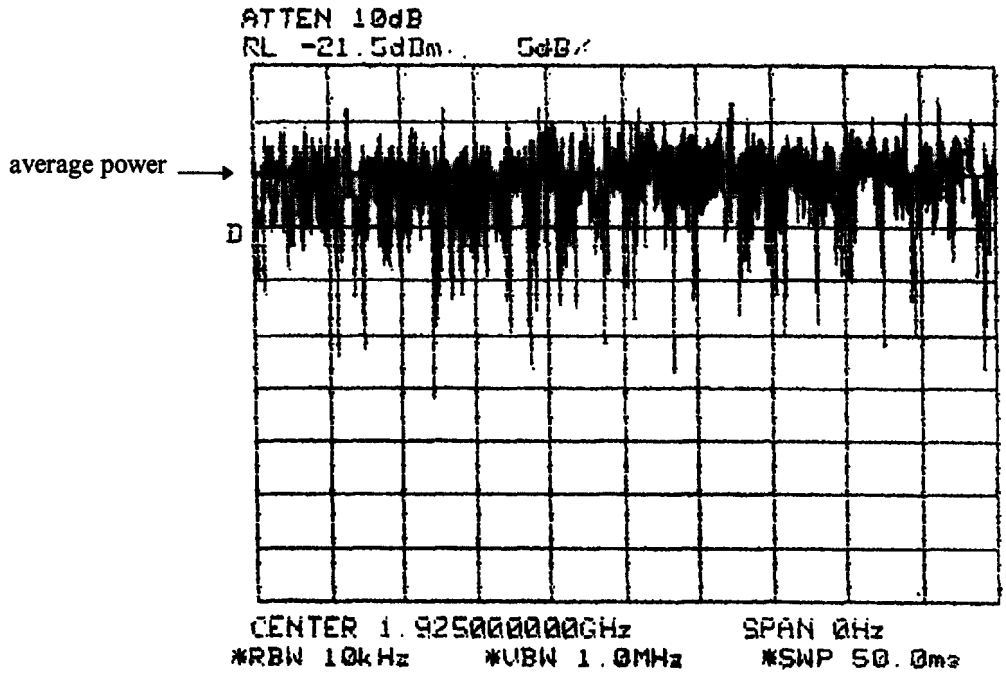

Figure 3: *Zero-span trace for same signal as in Fig. 2. The average power out of the resolution filter is –31.5 dBm.*

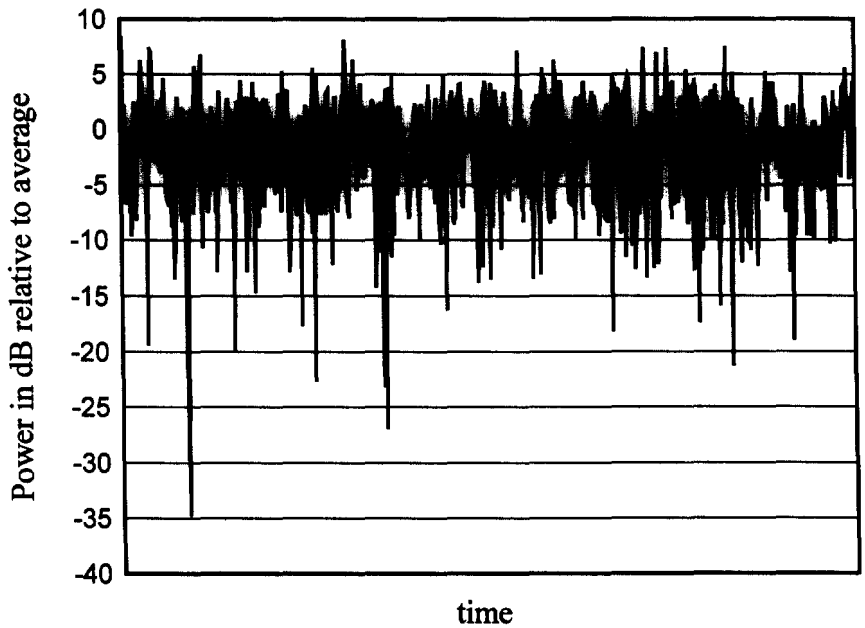

Figure 4: *Simulation results for 600 uncorrelated exponentially-distributed random variables.*

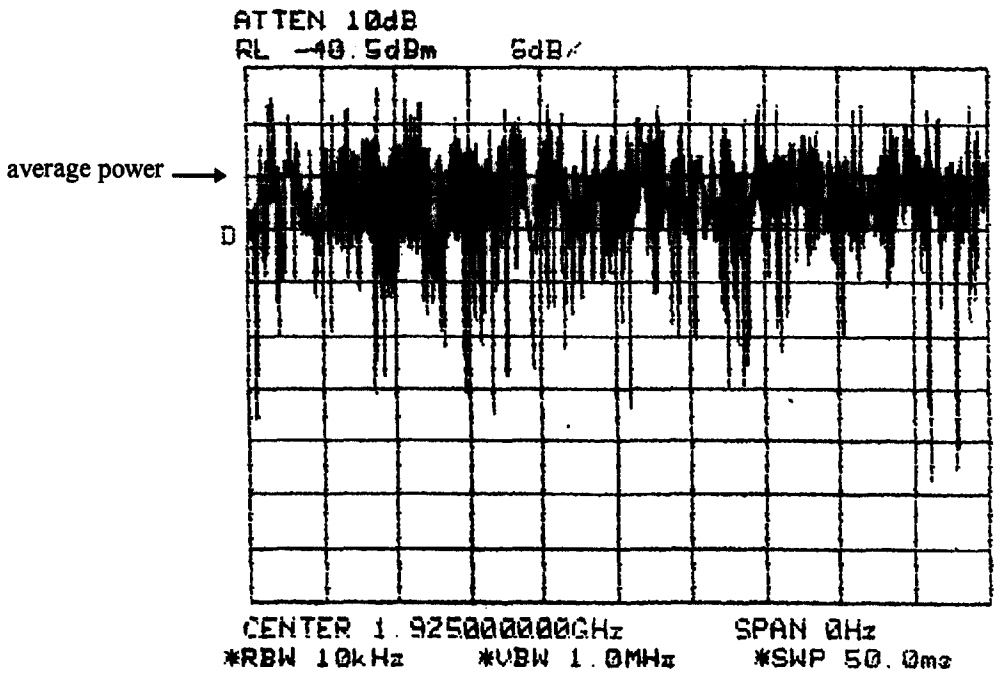

Figure 5: *Spectrum analyzer trace with input signal from noise generator. The average power out of the resolution filter is –50.5 dBm.*

To explore the properties of the filter output in more detail, peak-to-average ratios were investigated for various resolution bandwidths. Averages were measured using the built-in function, and compared to computed averages based on the filter bandwidth, the signal bandwidth, and the total power. It was found that "normal" detection gave results that were dependent on the sweep time, while sample detection gave results that were independent of sweep time. The computed average power was calculated according to:

$$P_{comp} = -20 + 10\log\frac{1.6B_{res}}{200}\ \text{dBm},$$

where B_{res} is the stated resolution bandwidth in kHz. The factor of 1.6 reflects the fact that, according to the analyzer user's manual, the effective noise bandwidth of the resolution filter is 1.6 times the stated bandwidth. The 200 on the denominator reflects the fact that the signal power is essentially confined to a 200-kHz bandwidth. For a 10-kHz resolution bandwidth, this formula yields –31 dBm, which only 2.5 dB above the measured value. Peak power was measured using the "max hold" setting of the analyzer.

The following table shows calculated average, measured average, measured peak, and peak-to-average ratio for a range of resolution bandwidths. In all cases, the video bandwidth was 1 MHz.

RBW	calc. avg.	meas. avg.	meas. peak	peak/avg.
1 kHz	–41.0 dBm	–40.8 dBm	–32.2 dBm	8.6 dB
3	–36.2	–36.2	–28.2	7.8
10	–31.0	–31.5	–24.3	6.2
30	–26.2	–27.0	–20.8	6.2
100		–22.7	–19.5	3.2
300		–20.8	–18.2	2.6
1000		–20.1	–17.3	2.8

For resolution bandwidths of 100 kHz and greater, the formula for calculating the average power is no longer accurate, so the corresponding entries in the table are blank. Note that when the entire signal falls within the resolution filter, the peak-to-average ratio is somewhat less than 3 dB, which is the theoretical value for modulation used. That is, the peak envelope power is 3 dB above the average power, which in this case is known to be about –20 dBm.

Video filtering was also investigated. This procedure, however, inherently averages the dB values of the signal. Using a 10 kHz resolution bandwidth, a 3 kHz video bandwidth, and the analyzer's video averaging function, the average power was –33.5 dBm, which is 2 dB below the true average. This is not surprising in light of (16), which shows that the difference would be 2.5 dB for exponentially-distributed power. This result suggests that the power out of the resolution filter is nearly exponentially-distributed.

CONCLUSIONS

It has been shown that when measuring the PSD of a signal using a resolution filter with a bandwidth significantly less than the signal bandwidth, the filter output power behaves very much like Gaussian noise. As a result, large variations in the filter output power can be expected. This explains why a single sweep on a spectrum analyzer exhibits the commonly-observed "roughness" or "fuzziness". The theoretical explanation of this phenomenon appears to be supported by the experimental results summarized here.

To obtain an accurate measurement of the PSD, and hence the potential for the device under test to interfere with other devices, it is necessary to perform some averaging on the resolution filter output. The use of the spectrum analyzer's "max hold" or "peak hold" function can cause a significant error, which will overstate the device's interference potential by many dB.

The analysis and experiment yield several other observations relevant to good measurement technique:

- "Sample" detection should be used to give accurate results, which are not affected by the sweep time used.

- The effective noise bandwidth of the analyzer's resolution filter will in general not be the same as the nominal resolution bandwidth. The instrument's documentation should be consulted to determine the proper correction factor, which should be taken into account in measuring the PSD.
- Averaging should be done on a linear, not a decibel, basis. Otherwise, the measured result will be up to 2.5 dB below the true average.

APPENDIX

SPECTRAL AND STATISTICAL PROPERTIES OF THE MEASURED POWER SPECTRAL DENSITY OF A STOCHASTIC SIGNAL OBSERVED OVER A FINITE TIME INTERVAL

It is often necessary to infer the spectral properties of an information-modulated carrier by observing the signal over some limited time interval. The power spectral density (PSD) is usually of particular interest, since it determines the potential of the signal to cause interference to a receiver of a given center frequency and bandwidth. To develop techniques that accurately measure the PSD, it is necessary to understand the relationship between the actual PSD and measured samples of the PSD. The purpose of this Appendix is to provide the analysis necessary for that understanding.

The information-bearing signal is modeled as a sample function of a stochastic (random) process. This Appendix develops derives the relationship between the PSD of this underlying stochastic process and the measured PSD, which is based on observation (using an instrument such as a spectrum analyzer) of a sample signal over some time interval of duration T, which typically is the integration time of the analyzer's resolution filter; T varies inversely with the filter bandwidth. It is first shown that the expected (average) value of the observed sample PSD is the actual PSD, provided T is sufficiently large compared to the signal correlation time (i.e., the filter bandwidth is small compared to the signal bandwidth). Next, it is shown that under this same condition, the observed PSD for a single-carrier waveform is a random variable that has a distribution which approaches the exponential distribution. The derivation is based on a signal for which the amplitude and phase are constant over a symbol interval, but in general change from symbol to symbol in a random manner. This idealized model is used to demonstrate the mechanism by which the narrowband filter power output is caused to be highly variable. A real system will depart somewhat from this model due to waveform shaping at baseband (to control spectral sidebands). Thus, the derivation of the measured PSD statistics must be considered a plausibility argument based on a particular family of modulation formats rather than an actual "proof". Exact results will obviously on the specific waveform processing (i.e., baseband pulse-shaping) in the transmitter, as well as on the properties of the measuring device. However, the mechanism demonstrated is generally applicable to single-carrier modulated signals, and the conclusion seems to be well-supported by experimental results, as discussed in the body of this paper.

Expected Value (Average) of the Observed PSD[4]

A signal $s(t)$ observed over the interval $[0, T]$ can be represented as:

$$s_T(t) = w_T(t)s(t) \tag{A-1}$$

[4]The analysis in this section is based on the general approach given by Papoulis, *op. cit.*, section 11-3.

where the observation window $w_T(t)$ is

$$w_T(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}. \tag{A-2}$$

The Fourier transform of $s_T(t)$ is

$$S_T(f) = \int_0^T s(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} w_T(t) s(t) e^{-j2\pi ft} dt \tag{A-3}$$

and the energy spectrum, assuming $s(t)$ is real, is

$$|S_T(f)|^2 = S_T(f) S_T^*(f) = \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} w_T(t_1) s(t_1) w_T(t_2) s(t_2) e^{-j2\pi f(t_1 - t_2)} dt_1 . \tag{A-4}$$

The expected value of the energy spectrum is

$$E\left[|S_T(f)|^2\right] = \int_{-\infty}^{\infty} w_T(t_2) e^{j2\pi ft_2} dt_2 \int_{-\infty}^{\infty} w_T(t_1) R_{ss}(t_1 - t_2) e^{-j2\pi ft_1} dt_1 \tag{A-5}$$

The inner integral is the Fourier transform of the transform of the autocorrelation $R_{ss}(\cdot)$ convolved (in the frequency domain) with $W_T(f)$. The transform of the autocorrelation term is:

$$\int_{-\infty}^{\infty} R_{ss}(t_1 - t_2) e^{-j2\pi ft_1} dt_1 = \int_{-\infty}^{\infty} R_{ss}(\tau) e^{-j2\pi f(\tau + t_2)} d\tau = \Phi_{ss}(f) e^{-j2\pi ft_2} \tag{A-6}$$

The inner integral therefore is

$$W_T(f) * \Phi_{ss}(f) e^{-j2\pi ft_2} = \int_{-\infty}^{\infty} W_T(f - \xi) \Phi_{ss}(\xi) e^{-j2\pi \xi t_2} d\xi \tag{A-7}$$

and (A-5) becomes:

$$E\left[|S_T(f)|^2\right] = \int_{-\infty}^{\infty} W_T(f - \xi) \Phi_{ss}(\xi) \int_{-\infty}^{\infty} w_T(t_2) e^{j2\pi t_2 (f - \xi)} dt_2 d\xi . \tag{A-8}$$

The inner integral in (A-8) is $W_T^*(f-\xi)$, giving:

$$E\left[\left|S_T(f)\right|^2\right] = \int_{-\infty}^{\infty} \left|W_T(f-\xi)\right|^2 \Phi_{ss}(\xi)d\xi. \tag{A-9}$$

With

$$W_T(f) = \int_0^T e^{-j2\pi ft}dt = \frac{\sin \pi fT}{\pi f} e^{-j\pi fT}, \tag{A-10}$$

(A-9) becomes:

$$E\left[\left|S_T(f)\right|^2\right] = \int_{-\infty}^{\infty} \frac{\sin^2 \pi(f-\xi)T}{\left[\pi(f-\xi)\right]^2} \Phi_{ss}(\xi)d\xi \tag{A-11}$$

The power per Hz at time T can be expressed as

$$\Phi_{sT}(f) = \frac{\partial}{\partial T}\left[\left|S_T(f)\right|^2\right] \tag{A-12}$$

and since differentiation is a linear operation, its expected value is:

$$E\left[\Phi_{sT}(f)\right] = \frac{\partial}{\partial T} E\left[\left|S_T(f)\right|^2\right] = \int_{-\infty}^{\infty} \frac{\sin 2\pi(f-\xi)T}{\pi(f-\xi)} \Phi_{ss}(\xi)d\xi = \Phi_{ss}(f) * \frac{\sin 2\pi fT}{\pi f}, \tag{A-13}$$

where $*$ is the convolution operator.

For a wideband information-bearing signal of bandwidth B_s, the autocorrelation $R_{ss}(\tau)$ will approach zero for $|\tau| > 1/B_s$. This means that if $T >> 1/B_s$, $\Phi_{ss}(f)$ is nearly constant over a frequency interval $1/T$ and (A-13) can be approximated as:

$$E\left[\Phi_{sT}(f)\right] \cong \Phi_{ss}(f) \int_{-\infty}^{\infty} \frac{\sin 2\pi(f-\xi)T}{\pi(f-\xi)} d\xi = \Phi_{ss}(f) \tag{A-14}$$

Thus, the average power from a filter with noise bandwidth $B_N << B_s$ centered on frequency f_0 will be $B_N \Phi_{ss}(f_0)$. The actual power from the filter at any given instant in time is a random variable. As shown below, its distribution approaches the exponential distribution as B_N / B_s shrinks.

It is worth noting that based on (A-13), $E\left[\Phi_{sT}(f)\right]$ can be expressed explicitly as a time-average. Since the inverse Fourier transform of $\sin 2\pi fT/\pi f$ is:

$$p_T(t) = \begin{cases} 1 & |t| \le T \\ 0 & |t| > T \end{cases}, \tag{A-15}$$

$E\left[\Phi_{sT}(f)\right]$ can be expressed as the Fourier transform of the autocorrelation of $\mathbf{s}(t)$ multiplied by $p_T(t)$:

$$E\left[\Phi_{sT}(f)\right] = \int_{-\infty}^{\infty} R_{\mathbf{ss}}(\tau) p_T(\tau) e^{-j2\pi f\tau} d\tau = \int_{-T}^{T} R_{\mathbf{ss}}(\tau) e^{-j2\pi f\tau} d\tau = E\left[\int_{-T}^{T} s(t)s(t+\tau)d\tau\right] \tag{A-16}$$

The argument of the expectation in (A-16) is exactly $\Psi_T(f,t)$ of eq. (4) in the body of this paper, so (A-15) serves to confirm eq. (5): that $E\left[\Psi_T(f,t)\right] = \Phi_{\mathbf{ss}}(f)$, provided T is sufficiently large compared to the correlation time of $\mathbf{s}(t)$.

Probability Distribution of the Observed PSD

In this section, a plausibility argument is presented to show why the observed PSD at a given frequency (or over a small frequency interval) is a random variable that tends to be exponentially distributed. Of necessity, assumptions must be made about the modulating function. The intent here is not to "prove" that the distribution tends toward a certain form, so much as to explain why it does.

The model used here applies to information-bearing signals which can be expressed in the form:

$$s(t) = a(t)\cos\left[2\pi f_c t + \theta(t)\right], \tag{A-17}$$

where $a(t)$ represents the signal envelope, f_c is the carrier frequency, and $\theta(t)$ is the phase offset. In general, both $a(t)$ and $\theta(t)$ are modulated by the transmitted information. To simplify the mechanics of the algebra, $s(t)$ will be expressed as:

$$s(t) = a(t)\,\mathrm{Re}\left\{e^{j(2\pi f_c t + \theta)}\right\} \tag{A-18}$$

where $\mathrm{Re}\{\cdot\}$ denotes the real part of the complex argument.

The modulating signal is assumed to consist of discrete information symbols, each of duration T_s. It will be assumed here that $a(t)$ and $\theta(t)$ are constant over each symbol

interval and are determined (according to the modulation format) by the value of the information symbol transmitted during that interval. This corresponds to a "square pulse" assumption.

The Fourier transform of $s(t)$, observed over the time interval [0, T] can be written as:

$$S_T(f) = \frac{1}{2}\int_0^T a(t)\left[e^{j[2\pi f_c t+\theta(t)]} + e^{-j[2\pi f_c t+\theta(t)]}\right]e^{-j2\pi ft}\,dt\,. \tag{A-19}$$

Since $a(t)$ and $\theta(t)$ are assumed constant over the interval $\left[nT_s,(n+1)T_s\right]$, $S_T(f)$ becomes:

$$S_T(f) = \frac{1}{2}\sum_{n=0}^{N-1} a_n \int_{nT_s}^{(n+1)T_s}\left[e^{j(2\pi f_c t+\theta_n)} + e^{-j(2\pi f_c t+\theta_n)}\right]e^{-j2\pi ft}\,dt\,, \tag{A-20}$$

where $N = T/T_s$, $a_n = a(nT_s)$, and $\theta_n = \theta(nT_s)$.

After some algebra, $S_T(f)$ can be cast in the form:

$$S_T(f) = \frac{1}{2}\sum_{n=0}^{N-1} a_n \left\{\frac{\sin\left[\pi(f_c - f)T_s\right]}{\pi(f_c - f)} e^{j\left[\theta_n+\pi(f_c-f)(2n+1)T_s\right]} + \frac{\sin\left[\pi(f_c + f)T_s\right]}{\pi(f_c + f)} e^{-j\left[\theta_n+\pi(f_c+f)(2n+1)T_s\right]}\right\} \tag{A-21}$$

Note that the real part of $S_T(f)$ is an even function of f and the imaginary part is an odd function of f. This is to be expected since $s(t)$ has been assumed real.

It is somewhat easier to understand the implications of (A-21) if $S_T(f)$ is translated to its equivalent baseband representation, which is the transform of the complex envelope of $s(t)$:

$$z(t) = a(t)e^{j\theta(t)}\,, \tag{A-22}$$

so that $s(t) = \mathrm{Re}\left\{z(t)e^{j2\pi f_c t}\right\}$. The baseband form of (A-21) is the Fourier transform of $z(t)$ observed over [0, T]:

$$Z_T(f) = \int_0^T z(t)e^{-j2\pi ft}dt = \frac{\sin \pi f T_s}{\pi f}\sum_{n=0}^{N-1} a_n e^{j\beta_n}, \tag{A-23}$$

where $\beta_n = \theta_n - \pi(2n+1)fT_s$. $Z_T(f)$ and $S_T(f)$ are related by:[5]

$$S_T(f) = \frac{1}{2}\left[Z_T(f - f_c) + Z_T^*(-f - f_c)\right] \tag{A-24}$$

Use of the baseband-equivalent form for analysis is valid for $f_c >> 1/T_s$ (i.e., the carrier frequency is much greater than the signal bandwidth), which normally is true for bandpass systems.

$Z_T(f)$ can be written as:

$$Z_T(f) = \frac{\sin \pi f T_s}{\pi f}(x + jy) \tag{A-25}$$

where

$$\begin{aligned} x &= \sum_{n=0}^{N-1} a_n \cos\beta_n \\ y &= \sum_{n=0}^{N-1} a_n \sin\beta_n \end{aligned} \tag{A-26}$$

With the $\{\beta_n\}$ independent and uniformly-distributed on $[0,2\pi)$, x and y are independent and zero-mean. By the Central Limit Theorem, as N grows, the distributions of x and y approach Gaussian. Their variances are $\sigma_x^2 = \sigma_y^2 = N \cdot E[a_n^2]/2$. If x and y are Gaussian, then $\gamma = x^2 + y^2$ is exponentially-distributed; that is, the probability density function (pdf) of γ is:

$$p(\gamma) = \frac{e^{-\gamma/\eta_\gamma}}{\eta_\gamma}, \tag{A-27}$$

where $\eta_\gamma = E[\gamma] = N \cdot E[a_n^2]$. Further, γ may be expressed as $\gamma = \eta_\gamma v$, where v is an exponential random variable with a mean of 1. The pdf of v is:

$$p(v) = e^{-v}. \tag{A-28}$$

[5]See *Digital Communications,* J. G. Proakis, third edition, McGraw-Hill, 1995, pp. 158-163.

The baseband-equivalent observed spectral energy density therefore may be expressed as:

$$\left|Z_T(f)\right|^2 = \frac{\sin^2 \pi f T_s}{(\pi f)^2} NE\left[a_n^2\right]v = \frac{\sin^2 \pi f T_s}{(\pi f)^2} \frac{T}{T_s} E\left[a_n^2\right]v \tag{A-29}$$

The observed PSD sample is:

$$\Phi_{zT}(f) = \frac{\partial}{\partial T}\left[\left|Z_T(f)\right|^2\right] = \frac{\sin^2 \pi f T_s}{(\pi f)^2 T_s} E\left[a_n^2\right]v \tag{A-30}$$

and its expected value is

$$E\left[\Phi_{zT}(f)\right] = \frac{\sin^2 \pi f T_s}{(\pi f)^2 T_s} E\left[a_n^2\right] . \tag{A-31}$$

As a sanity check, the total expected power in the signal $z(t)$ is:

$$\overline{P_z} = \int_{-\infty}^{\infty} E\left[\Phi_{zT}(f)\right]df = E\left[a_n^2\right] = E\left[a^2(t)\right] = E\left[\left|z(t)\right|^2\right], \tag{A-32}$$

as it should be.

Note that the average power in the baseband signal is twice that in the RF signal due to the way in which the baseband signal is defined; $z(t)$ is a complex envelope, whereas $s(t)$ is a real time-varying signal.

Output of The Spectrum Analyzer Resolution Filter

It is clear from (A-14), (A-24), (A-30), and (A-31) that

$$\lim_{T/T_s \to \infty} \Phi_{sT}(f) = v\Phi_{ss}(f); \tag{A-33}$$

that is, $\Phi_{sT}(f)$ is an exponentially-distributed random variable with a mean of $\Phi_{ss}(f)$. It therefore is reasonable to expect that the power output of the narrowband resolution filter of the spectrum analyzer will also be exponentially-distributed, with an average power which is the product of the filter noise bandwidth and the actual PSD $\Phi_{ss}(f)$.

The average power is easily derived. If the signal $s(t)$ is passed through a filter with transfer function $H(f)$, the output is a signal $g(t)$, which is a sample function of a random process $\mathbf{g}(t)$, that has a PSD given by:

$$\Phi_{gg}(f) = |H(f)|^2 \Phi_{ss}(f). \tag{A-34}$$

The average power at the filter output therefore is:

$$E[g^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 \Phi_{ss}(f) df. \tag{A-35}$$

If the bandwidth B_N of the filter is narrow relative to the bandwidth B_s of s(t), and the filter center frequency is f_0, then from the definition of B_N it follows that:

$$E[g^2(t)] \cong \Phi_{ss}(f_0) \int_{-\infty}^{\infty} |H(f)|^2 df = 2B_N |H(f_0)|^2 \Phi_{ss}(f_0), \tag{A-36}$$

where the factor of 2 accounts for the fact that $\Phi_{ss}(f)$ is the two-sided PSD.

To understand the statistical behavior of $g(t)$, is can be expressed as the convolution integral:

$$g(t) = \int_{-\infty}^{\infty} s(\tau) h(t-\tau) d\tau, \tag{A-37}$$

where $h(t)$ is the filter impulse response. A bandpass filter with a center frequency f_0 can be represented in terms of an equivalent baseband impulse response $h_b(t)$ as:

$$h(t) = h_b(t) e^{j2\pi f_0 t} + h_b^*(t) e^{-j2\pi f_0 t} = 2\,\mathrm{Re}\left\{ h_b(t) e^{j2\pi f_0 t} \right\}. \tag{A-38}$$

The filter impulse response will be limited to some duration T, which varies inversely with the filter bandwidth, and (A-37) becomes:

$$g(t) = 2\,\mathrm{Re}\left\{ e^{j2\pi f_0 t} \int_{t-T}^{t} s(\tau) h_b(t-\tau) e^{-j2\pi f_0 \tau} d\tau \right\}. \tag{A-39}$$

If $T >> 1/B_s$ the effect of the integral is again to sum many uncorrelated symbols, and the distribution of $g(t)$ will approach Gaussian.

As an example, if $h(t)$ is given by:

$$h(t) = \begin{cases} \frac{2}{T}\cos 2\pi f_0 t & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \tag{A-40}$$

then $h_b(t) = 1/T$ for $0 \le t \le T$ and 0 otherwise. The noise bandwidth of this filter is $B_N = 1/T$ and $\left|H(f_0)\right| = 1$. With this filter, $g(t)$ becomes:

$$g(t) = \frac{2}{T}\mathrm{Re}\left\{e^{j2\pi f_0 t} S_t(f_0)\right\}, \tag{A-41}$$

where

$$S_t(f) = \int_{t-T}^{t} s(\tau) e^{-j2\pi f\tau} d\tau, \tag{A-42}$$

which is simply a generalized version of (A-3).

For the signal format discussed in the previous section, it can be shown that:

$$\begin{aligned} S_t(f) = {} & \frac{1}{2} e^{-j2\pi(f-f_c)(t-T)} \frac{\sin\left[\pi(f_c - f)T_s\right]}{\pi(f_c - f)}\left[x_\alpha(t) + jy_\alpha(t)\right] \\ & + \frac{1}{2} e^{-j2\pi(f+f_c)(t-T)} \frac{\sin\left[\pi(f_c + f)T_s\right]}{\pi(f_c + f)}\left[x_\beta(t) + jy_\beta(t)\right] \end{aligned} \tag{A-43}$$

where $x_\alpha(t), y_\alpha(t), x_\beta(t)$, and $y_\beta(t)$ are each zero-mean random processes with mean-square values of $N \cdot E\left[a^2(t)\right]/2$ and each is the sum of N uncorrelated components. As $N = T/T_s$ becomes large, the distributions of $x_\alpha(t), y_\alpha(t), x_\beta(t)$, and $y_\beta(t)$ approach Gaussian.

Combining (A-41) and (A-43) gives:

$$\begin{aligned} g(t) = \frac{1}{T}\Bigg\{ & \frac{\sin\left[\pi(f_0 - f_c)T_s\right]}{\pi(f_0 - f_c)} r_\alpha(t)\cos\left[2\pi f_c t + \phi_\alpha(t)\right] \\ & + \frac{\sin\left[\pi(f_0 + f_c)T_s\right]}{\pi(f_0 + f_c)} r_\beta(t)\cos\left[2\pi f_c t - \phi_\beta(t)\right]\Bigg\} \end{aligned} \tag{A-44}$$

where $r_\alpha(t) = \sqrt{x_\alpha^2(t) + y_\alpha^2(t)}$ and $r_\beta(t) = \sqrt{x_\beta^2(t) + y_\beta^2(t)}$ represent the envelope variation of $g(t)$ and approach a Rayleigh distribution as N becomes large. The phase angles are:

$$\phi_\alpha(t) = \tan^{-1}\left[\frac{y_\alpha(t)}{x_\alpha(t)}\right] + 2\pi(f_0 - f_c)T \tag{A-45a}$$

$$\phi_\beta(t) = \tan^{-1}\left[\frac{y_\beta(t)}{x_\beta(t)}\right] + 2\pi(f_0 + f_c)T \tag{A-45b}$$

If the envelope of $g(t)$ is Rayleigh-distributed, the envelope power is exponentially-distributed. From (A-41),

$$E\left[g^2(t)\right] = \frac{2}{T^2} E\left[\left|S_t(f_0)\right|^2\right] \tag{A-46}$$

Since $\Phi_{ss}(f) = \frac{1}{T} E\left[\left|S_t(f)\right|^2\right]$ and $B_N = \frac{1}{T}$, it is clear that:

$$E\left[g^2(t)\right] = E\left[P_f(t)\right] = 2B_N \Phi_{ss}(f_0) \tag{A-47}$$

which agrees with the general expression of (A-36), as it should , since $\left|H(f_0)\right| = 1$ for the filter assumed in (A-40). The envelope power of the filter output therefore is:

$$P_f(t) = g^2(t) = 2B_N \Phi_{ss}(f) v(t), \tag{A-48}$$

where $v(t)$ is a unity-mean exponentially-distributed random variable. Again, the factor of 2 reflects the fact that a two-sided PSD has been used in the analysis for mathematical symmetry.

The correlation properties of the filter output signal are also of interest, for purposes of averaging uncorrelated samples to determine the PSD.

Letting $r(t) = \sqrt{v(t)}$ gives from (A-48):

$$g(t) = r(t)\sqrt{2B_N \Phi_{ss}(f_0)} \tag{A-49}$$