2/20/2015

COMMENTS ON THE INCENTIVE AUCTION DESIGN BY PROFS. TUOMAS SANDHOLM AND TRI-DUNG NGUYEN

To whom it may concern at the FCC,

We are submitting these comments in response to the FCC's public notice FCC 14-191, released on December 17th, 2014, that solicits comments on the incentive auction design (AU Docket No. 14-252, GN Docket No. 12-268).

We find that the FCC is proposing a very clever design for the incentive auction design overall. However, there are severe limitations in parts of the reverse auction design therein, both in the reverse auction structure and in the pricing rule. We propose a revised version of the reverse auction where the pricing is more flexible and informative to both sides of the market. We also propose a better methodology for decrementing prices across rounds of the reverse auction. It can be used in our reverse auction design or in the FCC's reverse auction design. We believe our designs are superior to the FCC's proposal, as will be detailed in the rest of this letter and in the attached two scientific papers. As background, Prof. Sandholm has fielded over 800 highly combinatorial reverse auctions.

The FCC put up for comment a descending clock auction (DCA) design that includes a price adjustment heuristic. That design is significantly more rigid than what we propose. Also, it does not take feasibility into account to nearly the same extent as our pricing technique does.

To our knowledge, no theory or experiments have been published so far to analyze the FCC's proposed design choices. In contrast, in the attached papers we show experiments on the performance of our design on real FCC interference data, and show that our design works very well indeed, both in terms of minimizing payment and avoiding the reverse auction terminating too early.

Our design differs from the FCC-proposed design in the following main ways:

- We propose sophisticated, scalable optimization for revising prices between rounds. Our pricing tries to minimize how much the FCC needs to pay out at the end, while the FCC-propose pricing heuristic does not accomplish that. Second, our pricing method takes into account repacking feasibility in a significantly better way than the pricing heuristic in the FCC proposal.
- Stations can move in any direction among their still-active options, at their discretion.
- Our design supports sophisticated control of the number of auction rounds.

Our proposal is described in detail in the first of the two attached papers.

Another confining—but potentially interesting—aspect in the FCC's proposed design is that the options are considered to form a "one-way" hierarchy. A bidder has to declare a preferred option (which is the option that he might get) at each point in the auction. A bidder is allowed to move the declared preferred option only downward in the hierarchy. So, a bidder can go from off-air to a lower band to an even lower band and then to accepting no offer, but not in the other direction. Also, a bidder is allowed to only move downward in the hierarchy from the option that she holds before the auction begins. We do not necessary agree that such a design is appropriate. However, if the FCC decides to move ahead with that restrictive design, our price optimization methodology can be beneficial even in that restrictive context. In particular, our Markov modeling and optimization techniques can be adapted to that setting as well, as described in Appendix A.1. of the first attached paper.

We will now briefly enumerate some of the serious downsides that the FCC-proposed design has, from which our design does not suffer.

1. **The prices offered to each station can only capture `local' feasibility information**. For each station, the FCC-proposed mechanism sets the discount factors R and r(t,s,b) according to the Vacancy V(t,s,b). The vacancy V(t,s,b) can only represent the `local' information of the station compared to its neighbors (by balancing among its neighbors their volumes and allocation feasibilities). This mean the prices offered to each station are based only on local information of that station. However, the repacking feasibility depends globally on stations in other parts of the network as well—something that our pricing methodology fully takes into account.

2. **It would be more informative for the auction system to receive responses from stations on non-preferred options as well**, that is, knowing that a station has already rejected a clock price for a non-preferred option would be informative in setting the prices of the remaining options. This is something that our descending clock auction proposal supports, but our pricing methodology can be used with or without this.

3. **The pricing rule does not take into account (the FCC's knowledge before and during the reverse auction of) the stations' valuation distributions** for the different options—or even the ranges that the valuations might take. For example, if the range of the valuations is relatively small and/or different valuations have ranges of different relative sizes, the FCC-proposed clock prices can—undesirably—get rejected right away. In setting the prices, the FCC-proposed framework does not take into account the expected payment that the FCC will pay to the stations. (Instead it sets prices at some discount rates derived from the `static' information about the station and its neighbors.) Our proposed design solves these problems.

4. <u>The price-setting rules described in Appendix D of FCC 14-191 allow stations to do **reverse engineering** to learn an undesirable amount of information about the mechanism that the FCC uses in setting the price trajectory and to **play games** against the FCC</u>. The prices offered to a station can reveal the parameters that the FCC is using and hence the station can make good guesses of the price trajectory that the FCC will offer in subsequent rounds. For example, given the prices offered after a few rounds, a station can do reverse engineering to find out the discount factor R. A station can also find out the reduction coefficients r(t,s,b) for the LV and UV bands. From these, the station can calculate/estimate the vacancy V(t,s,b) and hence have a good estimate of the response from neighboring stations so far. From these estimates, stations can make good estimates of the new prices that will be offered to them. With that information, stations may be able to play games against the FCC and thus the original motivation of using a descending clock auction for price discovery and for truthful bidding is compromised. Also, revenue is compromised.

5. <u>**The FCC-proposed bid processing provides significant and unfair advantage to stations that are being process first** (on the top of the queue).</u> This is because their assignments significantly limit the options of the remaining stations in the later part of the queue. Upon receiving bids, the FCC proposes to rank stations according to the attractiveness of their bids (either the intra-round prices in Section 4.3 or the benchmark over pricing in Section 4.4). Stations later in the queue are forced to take secondary options. Our design proposal avoids this problem. In computer science terms, the bid processing in the FCC proposal is a greedy algorithm while ours is optimization.

In our design, the system manages a sophisticated tradeoff between minimizing payment to the accepted bidders and repacking feasibility. Furthermore, the pricing affects the speed of the auction in terms of the number of rounds. Therefore, there is another tradeoff. On the one hand, if the offer prices are too high, many rounds are required, and that may be undesirable from the perspective of minimizing logistical effort. On the other hand, if the offer prices are too low, many bidders reject and the auction ends too quickly without properly serving its price-discovery purpose. Our optimization model for setting the offer prices balances between the expected payment, the repacking feasibility, and the speed of the auction. The model takes into account the estimated valuation distributions of the bidders (for all their options) and the interference characteristic of all stations together. This solves issues 1 and 3 listed above. In addition, our proposed process automatically considers all stations in the network in setting the offer prices to each station, so individual stations cannot estimate the price trajectory, thereby avoiding issue 4. Even with the inclusion of the hierarchy of options, our framework makes it easy to incorporate the valuable information available from the fact that a station rejects an option, even if that is not the preferred option, which solves issue 2. Our design also solves issue 5, as explained above.

Please do not hesitate to contact Prof. Sandholm (sandholm@cs.cmu.edu) if you would like any further information or if we can be of any other assistance.

Sincerely,

Dr. Tuomas Sandholm and Dr. Tri-Dung Nguyen
Professors

(We are making these comments as individuals, and the views are ours. The views may not necessarily represent the views of our universities.)

ATTACHMENTS

Two scientific papers:

- Multi-Option Descending Clock Auction
- Optimizing Prices in Descending Clock Auctions

Multi-Option Descending Clock Auction*

Tri-Dung Nguyen† Tuomas Sandholm ‡

Abstract

A descending clock auction (DCA) is a mechanism for buying items from multiple sellers. The auctioneer starts by offering bidders high prices and gradually decreases the prices while there is competition. The academic literature has focused on the vanilla case where each bidder has two options: to accept or reject the offered price. However, in many settings—such as the FCC's imminent incentive auction—each bidder may be able to sell one from a *set of options*. We present a multi-option DCA (MDCA) framework where at each round, the auctioneer offers each bidder different prices for different options, and a bidder may find multiple options still acceptable. A key component is the technique for deciding how to set prices during the MDCA. This is significantly more difficult in an MDCA than in a DCA. We develop a Markov chain model for representing the dynamics of each bidder's state (which options are still acceptable), as well as an optimization model and technique for finding prices to offer to the different bidders for the different options in each round—using the Markov chain. The optimization minimizes total payment while ensuring feasibility in a stochastic sense. We also introduce percentile-based approaches to decrementing prices. Experiments with real FCC incentive auction interference constraint data reveal that the optimization-based approach dramatically outperforms the simple percentile-based approach both under symmetric and asymmetric bidder valuation distributions—because it takes feasibility into account in pricing. Both techniques scale to the large.

1 Introduction

A *descending clock auction (DCA)* is a mechanism for buying items from multiple potential sellers. A vanilla DCA works as follows, and has remarkably strong incentive properties [18]. Consider the following setting where the auctioneer wants to buy items. Each seller $i \in N$ has a specific type of item and decides to sell it or not depending on the offer price. The items from the sellers could be substitutable and complementary to the buyer. The auctioneer has a target number of items to buy, T, and there is a feasibility function $F : 2^N \to \{0, 1\}$ that specifies, for each subset of potential sellers, S, whether the items from S can fulfill the target T or not, that is, $F(S) = 1$ if the combined items fulfill the target. A simple example of this is the case where the sellers have identical items and the auctioneer wants to buy a target number, T, of them. In that case, the feasibility function is simply $F(S) = 1$ if $|S| \geq T$, and $F(S) = 0$ otherwise. In real-world applications—such as the FCC spectrum reverse auctions discussed below—the feasibility function can be highly complex. Often it cannot be given in closed form, but rather is stated through constraints as an optimization problem.

In the vanilla DCA, the auctioneer sends offer prices to the sellers and checks whether they accept those prices. Bidders who accept the offers are called *active*. If the combined items from the active bidders fulfill the target, then the auctioneer reduces the prices further in the next round and repeats the process. If at some point the items from the active bidders do not fulfill the target, then the auctioneer goes back to the last step and conducts a last-round adjustment to offer higher prices to some declined bidders so that feasibility is obtained.

*Patent pending.

†University of Southampton, School of Mathematics.

‡Carnegie Mellon University, Computer Science Department; Corresponding author email: sandholm@cs.cmu.edu.

The DCA framework is agnostic to how offered prices are decremented across rounds. Doing that well is a key problem for which no solutions had been published until recently. Nguyen and Sandholm [19] presented techniques for this.

In their *percentile-based* family of techniques, the approach is to set the offer price at some fixed percentile of the (buyer's model of the) distribution of that bidder's valuation. For example, the prices could be set so that each bidder has the same probability of accepting her offer. The choice of the percentile would depend on what the auctioneer aims for on the trajectory of the sizes of sets of active bidders through the rounds. For example, the trajectory could be set so that the expected number of rejections in each round is distributed evenly throughout the auction. Another example of a trajectory is to set a fixed percentage of rejection in each round, that is, the expected number of rejections would be proportional to the size of the remaining set of active bidders.

Those methods have several drawbacks. First, having a fixed percentile means there is no way to distinguish bidders with greater influence on the feasibility function; hence the final payment will likely be unnecessarily high due to the probabilistic inclusion of high-priced bidders. More importantly, those methods do not have any special treatment for the degree of interaction among the items in the feasibility function.

Nguyen and Sandholm [19] also presented an optimization model for setting the prices in the vanilla DCA. The model is designed to minimize the expected final payment while ensuring feasibility in a stochastic sense. It is flexible in that it can incorporate bidder-specific characteristics with respect to feasibility.

That paper—and, to our knowledge, all other papers on incentive auctions and on the DCA to date [18, 21]—consider the setting where bidders have only two options, that is, either to sell or not.

In contrast, in many settings, each seller may be able to sell *one from a set of options* to the auctioneer. The DCA can be generalized to this setting by offering each bidder a separate price for each of her options in each round. However, the problem of decreasing prices appropriately during the DCA is drastically more intricate in this *multi-option DCA (MDCA)* setting.

We present an MDCA framework and price-decrementing techniques for it. The model captures a broad set of applications, including the imminent flagship application of DCAs, the FCC incentive auctions—where an MDCA will be needed and used.

1.1 Incentive auctions

The FCC has been selling radio spectrum licenses via auctions since 1994 [2, 15]—in recent years via combinatorial auctions [3, 4]. However, there is not enough spectrum left to sell for the new high-value spectrum uses that have arisen. The idea of *incentive auctions*, therefore, is to buy some of the existing licenses back from their current holders, which frees up spectrum, and then to sell spectrum to higher-value users. The idea of such incentive auctions was introduced in the 2010 National Broadband Plan [6]. It is motivated by the fact that the demand and the value of over-the-air broadcast television has been declining while the demand for mobile broadband and wireless services has increased dramatically in recent years. Given the limited spectrum resources, incentive auctions were introduced as a voluntary, market-based means of repurposing spectrum. This is done by creating a market that exchanges the usage rights among the two groups of users: (a) existing TV broadcasters and (b) wireless broadband networks. Three key players in this market are existing spectrum owners, spectrum buyers, and the FCC, which acts as the intermediator.

An incentive auction consists of three stages [11] (see also a whitepaper about design choices by Hazlett et al. [12]):

1. *Reverse auction*: some spectrum currently used by TV broadcasters is bought back.[1, 2]

2. *Repacking*: remaining broadcasters are reallocated to a smaller spectrum band.

3. *Forward auction*: freed spectrums is sold via a (combinatorial) auction for use in wireless broadband networks.[3]

In the reverse auction, we need to find a set of stations to be reallocated to lower-band channels and a separate set of stations to be bought off the air, in order to achieve the following goals: (a) meet some target on the number of contiguous channels freed on the higher spectrum band and (b) minimize total payment by the FCC. The FCC is required to respect the broadcasters' carry-right, which means, in the context of a DCA, that a station that rejects the offer still has the right to stay on the air, but possibly on a lower spectrum band. This repacking stage needs to ensure that all the stations that rejected their offers can be feasibly repacked into the allocated band without violating the engineering constraints, that is, interference-free population coverage, as we detail later in the paper.

There are $n = 2177$ stations and $m = 49$ channels (ranging from channels 2 to 51, with channel 37 not available). The channels are divided into two bands: the very-high frequency (VHF) band (54-216 MHz) and the ultra-high frequency (UHF) band (614-698 MHz) [6]. These correspond to VHF channels 2-13 and UHF channels 14-51. The VHF band is further divided into two bands: lower VHF (LVH) with channels 2-6 and upper VHF (UVH) with channels 7-13.

The aim of the reverse auction is to clear a number of channels in the high-frequency band, say channels 33-51. This means all stations that are currently in this band need to either go off-air or be reallocated to lower-frequency channels 2-32. Stations in channels 2-32 could alternatively go off-air or be reallocated to different channels. It is these options that beget the need for a *multi-option* DCA, as we detail in the next section.

At each round of the DCA, the repacking problem needs to be solved in order to check whether the remaining stations can be feasibly reassigned to the targeted lower-band channels. Other groups have recently also tackled the repacking part (e.g., Leyton-Brown [14]). There are a large number—2.9×10^6—of engineering interference constraints requiring pairs of stations not to be allocated in the same or adjacent channels. Also, some stations are restricted to being allocatable to only a subset of the channels. The FCC has published all these engineering constraints on the FCC web site [9], and we use these real constraints in our experiments, as detailed later.

The FCC announced in June 2014 that a multi-option DCA will be used for the reverse auction [8], but left open at that point the important question of how prices will be decremented across rounds. Our paper studies pricing techniques for multi-option DCAs. We present general techniques, both ones based on percentiles and ones based on optimization. We present experiments using real interference constraints from the imminent FCC incentive auction.

[1] The FCC has decided to use some form of DCA for the reverse auction instead of a VCG mechanism because 1) the VCG winner determination problem with the interference constraints is prohibitively complex [17] (and would have to be solved $|N| + 1$ times to obtain VCG prices also), and 2) a small approximation error in solving can lead to significant over-payment [16].

[2] Combinatorial reverse auctions are used extensively for sourcing goods and services in industry and government (e.g., [25, 20]). Typically, pay-your-winning-bids (i.e., first-price) pricing is used. Usually, 1-to-3 rounds of bidding are used, with feedback to bidders between rounds. Also, continuous variants have been used (when the number of items in the auction is small), where tentative winner determination is conducted each time a bid is submitted or revised [25]. Feedback and bidder strategies in combinatorial auctions have also been studied in laboratory experiments [1]. Combinatorial reverse auctions have been proposed for sourcing carrier-of-last-resort responsibility for universal service [13]. In contrast to combinatorial reverse auctions, in the DCA, prices are non-combinatorial: only individual items are priced.

[3] Once the reverse auction phase is completed and the remaining stations are repacked, the FCC announces the cleared spectrum that is now available for purchase. Buyers then submit bids on bundles of licenses. The FCC solves a winner determination problem to decide which bids to accept. This does not involve the engineering constraints, so it resembles a standard combinatorial auction. Existing algorithms (e.g., [23, 24, 25, 26, 5, 22]) can be used for this. (The FCC may iterate between the reverse and forward auctions to try to (approximately) equilibrate supply and demand before actual purchases and sales are made [10].)

Very recently, in December 2014 (in parallel with our research) the FCC put up for comment an MDCA design that includes a price adjustment heuristic [10]. It is much more rigid than what we propose. Also, it does not take feasibility into account to nearly the same extent as our pricing technique does. To our knowledge, no theory or experiments have been published so far to analyze the FCC-proposed design choices. In Appendix A.1 we discuss how our techniques can be of benefit even within the confines of that proposal.

2 Multi-option descending clock auction (MDCA)

We present our MDCA in the domain of the FCC incentive auction, where the number of options per bidder is at most three (plus the option of rejecting all three options), but the techniques can be directly extended to any number of options per bidder.

In the FCC setting, bidders (stations) that are currently in the UHF band, have four choices[4]: go off-air, go down to LVH (lower VHF), go down to UVH (upper VHF), or reject all of these options. Stations currently in the UVH band, have three choices: go off-air, go down to LVH (lower VHF), or reject both of these options. Stations currently in the LVH band have two choices: go off-air or reject. When a station rejects all options, it will be (re)allocated to a channel in its original band (without any payment).

We denote the set of options $D = \{\text{OFF, LVH, UVH}\}$ with indices $k \in \{1, 2, 3\}$, respectively. D_i is the set of options that station i can choose. $D_i = \{1, 2, 3\}$ for station in UHF, $D_i = \{1, 2\}$ for stations in UVH, and $D_i = \{1\}$ for stations in LVH. We denote by v_{ik} be the valuation of station i for option k, that is, the price at which the station is indifferent between accepting the option and rejecting all options.

For each station $i \in \mathcal{N}$, let $\mathcal{A}_i^{(r)} \in \{0, 1\}^3$ be the binary vector that indicates whether, at round r, station i is still active for the three participation options. That is, $\mathcal{A}_{i1}^{(r)} = 1$ if station i is still active to go off-air and $\mathcal{A}_{i1}^{(r)} = 0$ otherwise. Similarly, $\mathcal{A}_{i2}^{(r)} = 1$ ($\mathcal{A}_{i3}^{(r)} = 1$) if station i is still active to be downgraded to LVH (UVH). (No upgrading to higher bands is allowed so $\mathcal{A}_{i2}^{(r)} = 1$ is possible only if station i is currently in the UHF band and $\mathcal{A}_{i1}^{(r)} = 1$ is possible only if the station is currently in either the UHF or the ULV band.) Figure 1 shows the bidders' options and the corresponding possible allocations. The first row shows which band the stations currently belong to while the first column shows 8 possible scenario for $\mathcal{A}_i^{(r)}$. The second row includes 4 possible outcomes of the final allocation, i.e., OFF (off-air), LVH, UVH, or UHF (reject all offers and stay in UHF). For each of the 8 rows, cells that are marked with a cross X are possible allocations. Because upgrading is not allowed, some cells are colored in gray. Empty cells are not applicable due to the corresponding choices of $\mathcal{A}_i^{(r)}$.

	LVH Stations				UVH Stations				UHF Stations			
$A_i^{(r)}$	OFF	LVH	UVH	UHF	OFF	LVH	UVH	UHF	OFF	LVH	UVH	UHF
0,0,0		X					X					X
0,1,0						X	X			X		X
0,0,1											X	X
0,1,1										X	X	X
1,0,0	X	X			X		X		X			X
1,1,0					X	X	X		X	X		X
1,0,1									X		X	X
1,1,1									X	X	X	X

Figure 1: Bidders' options and allocation possibilities.

In the beginning, we can assume that all the bidders are active for all their available options. That is, in the first round $r = 0$, $\mathcal{A}_i^{(r)} = \iota(D_i)$ which is the indicator vector in $\{0, 1\}^3$ with non-zero indices in D_i.

[4] The option of stations sharing channels is not included as an option in the auction—by the FCC or by us—as it can be viewed as one station going off-air and the other bidding in the DCA.

Stations in UHF will be offered three prices. Stations in UVH and LVH will be offered two prices and one price, respectively. We denote by $p_{ik},\ k \in \{1,2,3\}$, the offer price to station i for option k.

In each DCA round, the auctioneer offers each bidder prices for all the options for which the bidder is still active. The bidder then evaluates them and decides which of those options are still acceptable. As long as a bidder is still active for an option, the bidder enters the next round where the same process will be repeated for the remaining active options. If a bidder becomes inactive for all options, that station needs to be allocated to its current band without payment.

ALGORITHM 1: A Multi-Option DCA Framework

Input: A set of stations $\mathcal{N} = \{1,\dots,n\}$, an auctioneer with a feasibility function $F\ :\ \prod_{i\in\mathcal{N}} \mathcal{A}_i \to \{0,1\}$. A target number of rounds allowed m. Initial valuation function estimates $v_{ik},\ k \in D_i$.

Output: For each station owner i, a feasible final set of active options $\mathcal{A}_i$, i.e. $F(\mathcal{A}) = 1$, the corresponding offer price vector $\boldsymbol{p}$ for achieving $\mathcal{A}$, and the final assignment of stations to their options to minimize the expected payment.

1. Set the initial prices $\boldsymbol{p}$ at the reserves. Set $r = 0$ and let $\mathcal{A}_i^{(r)} = \iota(D_i)$;

2.**for** *round* $r = 1...m$ **do**

  2.1. Find a vector of prices $\boldsymbol{p}$ to offer the bidders on their active options;

  2.2. Update the sets $\mathcal{A}^{(r)}$ as follows: $\mathcal{A}_{ik}^{(r)} = 1$ only if option k is still available for station i, i.e. $\mathcal{A}_{ik}^{(r-1)} = 1$, and if bidder i accepts price p_{ik}, i.e. $p_{ik} \geq v_{ik}$;

  if *bidders' options according to* $\mathcal{A}^{(r)}$ *still leads to feasible packing, i.e.* $F(\mathcal{A}^{(r)}) = 1$*,* **then**

    2.2.1. Update the distributions of the bidders' valuations;

  else

    2.2.2. Reset the sets $\mathcal{A}^{(r)}$ to those in the previous round and enter Step 3;

3. Final round adjustment to find the winners and their prices;

There are two missing pieces in Algorithm 1, which are to find the offer prices in step 2.1 and to check the repacking feasibility within step 2.2. We will now discuss these pieces, starting from the latter.

2.1 Repacking feasibility problem

Let $\mathcal{S}$ be a set of stations that needs to be repacked into a set of channels $\mathcal{C}$. We use i and j as indices for stations and k as indices for channels. Let $\mathcal{C}_i \subset \mathcal{C},\ i \in \mathcal{S}$, be the set of feasible channels for station i. Let $\mathcal{I}_c$ be the list of triplets (i,j,k) such that stations i and j cannot be assigned to the same channel k. Let $\mathcal{I}_a$ be the list of triplets (i,j,k) such that stations (i,j) cannot be assigned to channels k and $k+1$, respectively. Data for $\mathcal{C}_i$, $\mathcal{I}_c$ and $\mathcal{I}_a$ are available from the domain file and the interference-paired file on the FCC web site [9], which we use in our experiments.

Let $F\ :\ \prod_{i\in\mathcal{N}} \mathcal{A}_i \to \{0,1\}$ be the feasibility function

$$F(\mathcal{A}) = \begin{cases} 1, & \text{if } \mathcal{P}(\mathcal{A},\mathcal{C}) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{P}(\mathcal{A},\mathcal{C})$ is the set of feasible assignments of stations in $\mathcal{A}$ to available channels $\mathcal{C}$:

$$\mathcal{P}(\mathcal{A},\mathcal{C}) \quad = \quad \left\{ \boldsymbol{z} : \quad \begin{array}{l} z_{ik} \in \{0,1\}, \forall i \in \mathcal{N} \text{ and } k \in C_i, \\ \sum_{k \in C_i} z_{ik} \geq 1 - \mathcal{A}_{i1}^{(r)}, \forall i \in \mathcal{N}, \\ \sum_{k \in LVH} z_{ik} \leq \mathcal{A}_{i2}^{(r)}, \forall i \notin LVH, \\ \sum_{k \in UVH} z_{ik} \leq \mathcal{A}_{i3}^{(r)}, \forall i \notin UVH, \\ z_{ik} + z_{jk} \leq 1, \forall (i,j,k) \in \mathcal{I}_c, \\ z_{ik} + z_{jk+1} \leq 1, \forall (i,j,k) \in \mathcal{I}_a, \end{array} \right\} \tag{1}$$

Here, z_{ik} is a binary variable that indicates whether station i is assigned to channel k. The second set of constraints requires that, if a station decided not to go off-air ($\mathcal{A}_{i1}^{(r)} = 0$), the station needs to be (re)allocated to reserve its carry-right. The third set of constraints requires that, for stations currently not in LVH and that do not accept to move to LVH, they will not be allocated to LVH. The fourth set of constraints enforces the analogous requirement for the UVH band. The last two sets of constraints ensure that the allocation avoids interference. Later in the paper we present an optimization technique for decrementing prices that incorporates this feasibility problem.

3 Setting offer prices

A key component of a DCA is how the prices offered to active bidders are decremented across rounds. The auctioneer needs to consider the tradeoff between minimizing payment to the accepted bidders and fulfillment of the target (repacking feasibility in the case of incentive auctions).

Furthermore, the pricing affects the number of rounds the auction takes. This begets another tradeoff. If the prices decrease too slowly, many rounds are required, and that may be undesirable from the perspective of minimizing logistical effort. If the prices decrease too quickly, many bidders reject and the auction ends without properly serving its price-discovery purpose.

How the prices are changed across rounds should depend on (a) the estimated value functions of the bidders, (b) the importance of the items for the target to be fulfilled (including interference in the case of incentive auctions), and (c) the desired number of rounds. We provide an optimization model that incorporates these considerations.

3.1 A stochastic program for optimizing offer prices

We first present a stochastic program for finding optimal offer prices. Although we will not attempt to solve this model due to its complexity, it provides us an understanding of the uncertainty involved. For each station i and for each option $k \in \{1, 2, 3\}$, suppose the valuations v_{ik} are drawn from some distribution on support $[l_{ik}, u_{ik}]$. Assume that the auctioneer knows these distributions but not the valuations.

Let $X_{ik}(p_{ik})$ be the Bernoulli random variable that indicates whether bidder i will accept the offer for option k at price p_{ik}. Let $\mathbf{X} = (X_{ik}), i \in \mathcal{N}, k \in \{1, 2, 3\}$ be one such matrix of realized outcomes. We denote by $Q(\boldsymbol{p})$ the probability distribution over $\mathbf{X}$. Given outcome $\mathbf{X}$, the auctioneer faces a problem of choosing which active options from each station to choose so as to minimize total payment while achieving a feasible assignment. This can be formulated as the integer program

$$f(\boldsymbol{p}, \mathbf{X}) = \min_{\boldsymbol{z}} \sum_{i \in \mathcal{N}} \sum_{k \in X_i} z_{ik} p_{i,o(k)} \text{ s.t. } \mathbf{z} \in \mathcal{P}(\mathbf{X}, \mathcal{C});\ \text{See (1) for a formulation of } \mathcal{P}(\cdot, \cdot). \tag{2}$$

Here $\boldsymbol{z}$ is the assignment and $o(k)$ is the band that channel k is in. If the current channel of station i and channel k belong to the same band, we can assume $o(k) = 0$ with $p_{i,0} = 0$. The set of constraints here is similar to those in repacking feasibility (1). The only difference is that we have replaced $\mathcal{A}^{(r)}$ with $\mathbf{X}$. Note that the repacking problem might be infeasible. In that case, we define $f(\boldsymbol{p}, \mathbf{X}) = +\infty$.

Given that $\mathbf{X}$ is a random matrix generated by Bernoulli random variables with success rates dependent on $\boldsymbol{p}$, $f(\boldsymbol{p}, \mathbf{X})$ is also a random variable. A natural approach in stochastic programming is to find the optimal offer price $\boldsymbol{p}$ such that the expected total payment is minimized:

$$\min_{\boldsymbol{p}} \quad E_{\mathbf{X} \sim Q(\boldsymbol{p})} [f(\boldsymbol{p}, X)]. \tag{3}$$

This is very difficult to solve, if not impossible, because even a functional evaluation of the term $f(\boldsymbol{p}, \mathbf{X})$ inside the expectation for a given $\boldsymbol{p}$ and a realization $\mathbf{X}$ involves solving a large-scale integer program that is very difficult to solve to optimality. (This problem is actually a generalized graph coloring problem and its IP formulation is of the same form as the final settlement problem described later in Section 4.) We will design a deterministic approximation of (3). One major difficulty in designing a deterministic program to approximate the stochastic program (3) lies in the complexity caused by the multiple possible states that a station can be in throughout the DCA. To develop a percentile-based model or a deterministic optimization model, we need to model the constraints in relation to the decision variable $\boldsymbol{p}$. This requires a mathematical representation for the (expected) number of stations that will finally end up in each band—which is non-trivial: these will be highly nonlinear functions of $\boldsymbol{p}$.

To elaborate, consider a station that is in the UHF band and has all three options still active. Suppose the auctioneer offers prices to the options in such a way that the acceptance probabilities are q_1, q_2, q_3. Now, what is the probability of the station ending up in the UHF band (i.e., rejecting all offers) after m rounds? What about the other bands? We propose using Markov chains to model the dynamic of the station's state throughout the DCA.

3.2 Modeling the dynamics of a station in the DCA using a Markov chain

We denote by q_{ik} the probability that station i finds price p_{ik} acceptable for option k. For example, if the valuation distribution is uniform, $q_{ik} = \frac{u_{ik}-p_{ik}}{u_{ik}-l_{ik}}$. For convenience, we regard $\boldsymbol{q}$ as the decision variables that the auctioneer has to set.

Consider a station that is currently in the UHF band. Suppose at the current round r, the auctioneer offers acceptance rates (q_1, q_2, q_3) to the three available options. What is the station's probability distribution over its states in the next round? How about after m rounds if the acceptance rates in each round are kept the same at (q_1, q_2, q_3)? To answer these questions, we need to understand the state evolution of each station. We will use Markov chains for this.

We define the state of a station to depend on which of the three options are still active. There are $2^3 = 8$ possible states formed by the power set of {OFF, LVH, UHF}:

$S_1 = \{\text{OFF, LVH, UVH}\}$	$S_2 = \{\text{OFF, LVH}\}$	$S_3 = \{\text{OFF, UVH}\}$	$S_4 = \{\text{LVH, UVH}\}$
$S_5 = \{\text{OFF}\}$	$S_6 = \{\text{LVH}\}$	$S_7 = \{\text{UVH}\}$	$S_8 = \{\text{None}\}$

Table 1: States of a station with respect to active options.

Theorem 1 *The state transition of the m-round DCA with fixed acceptance probabilities (q_1, q_2, q_3) is equivalent to a single-round DCA with acceptance probabilities q_1^m, q_2^m, q_3^m.*

We now present an exposition of this result, which also serves as the proof.

Figure 2 (a) shows the Markov chain for a station currently in the UHF band. For example, the transition probability to go from S_1 to S_1 is $q_1q_2q_3$ since this occurs when all three offers are acceptable. We can derive the transition probabilities between the remaining states analogously.

We denote by Γ_{UHF} the corresponding transition matrix

$$\begin{bmatrix} q_1q_2q_3 & q_1q_2(1-q_3) & q_1(1-q_2)q_3 & (1-q_1)q_2q_3 & q_1(1-q_2)(1-q_3) & (1-q_1)q_2(1-q_3) & (1-q_1)(1-q_2)q_3 & (1-q_1)(1-q_2)(1-q_3) \\ 0 & q_1q_2 & 0 & 0 & q_1(1-q_2) & (1-q_1)q_2 & 0 & (1-q_1)(1-q_2) \\ 0 & 0 & q_1q_3 & 0 & q_1(1-q_3) & 0 & (1-q_1)q_3 & (1-q_1)(1-q_3) \\ 0 & 0 & 0 & q_2q_3 & 0 & q_2(1-q_3) & (1-q_2)q_3 & (1-q_2)(1-q_3) \\ 0 & 0 & 0 & 0 & q_1 & 0 & 0 & (1-q_1) \\ 0 & 0 & 0 & 0 & 0 & q_2 & 0 & (1-q_2) \\ 0 & 0 & 0 & 0 & 0 & 0 & q_3 & (1-q_3) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We denote by $I_{UHF,i}^{(r)} \in \mathbb{R}^8$ the probability vector of station i being in state $S_1, \ldots, S_8$. At the beginning of the auction, $I_{UHF,i}^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$. By the end of round m,

$$I_{UHF,i}^{(m)} = \Gamma_{UHF} I_{UHF,i}^{(m-1)} = \cdots = \Gamma_{UHF}^m I_{UHF,i}^{(0)}$$

The formulation for E_{UHF}, E_{UVH} and E_{LVH} involves Γ_{UHF}^m which is a polynomial of q_1, q_2, q_3 and of order $3m$. We need a fast way to evaluate this. It turns out that the eigenvalues of Γ_{UHF} have special forms of

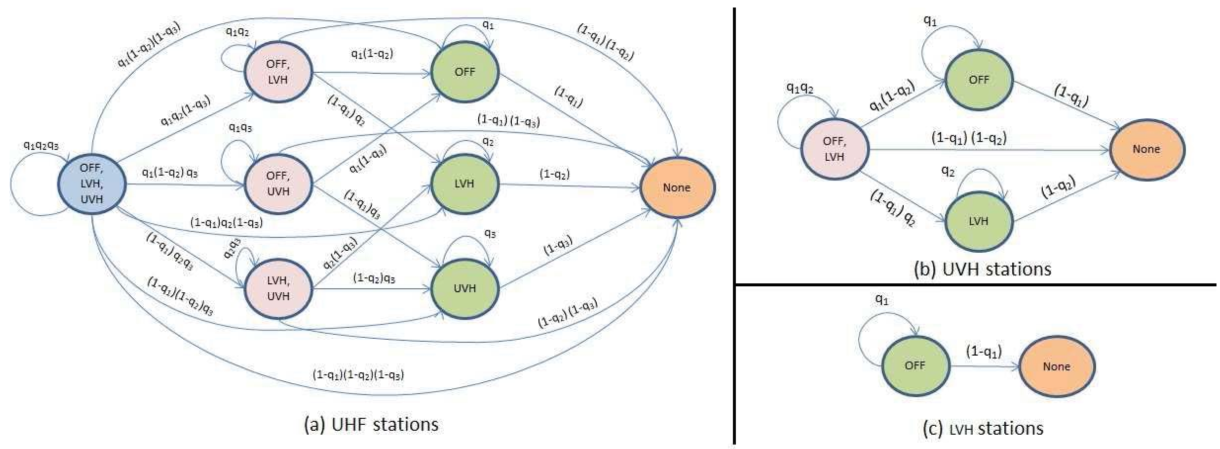


Figure 2: Markov chain for a station currently in UHF (a), UVH (b), and LVH (c).

$\prod_{j \in S} q_j$ for each of the 2^3 subset $S \subset \{1, 2, 3\}$. Let Λ denote the diagonal matrix with these eigenvalues on the diagonal and let V be the matrix containing the corresponding eigenvectors on its columns:

$$\Lambda = diag\left(\begin{bmatrix} 1 \\ q_1 \\ q_2 \\ q_3 \\ q_1 q_2 \\ q_1 q_3 \\ q_2 q_3 \\ q_1 q_2 q_3 \end{bmatrix}\right) \quad \text{and} \quad V = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We observe that the eigenvectors do not depend on q_i and the eigenvalues have a nice form as a function of q_i. Thus we can utilize the single value decomposition for Γ_{UHF} as $\Gamma_{UHF} = V \Lambda V^{-1}$. From that, we have $\Gamma^m_{UHF} = V \Lambda^m V^{-1}$. By defining $\kappa_i = q_i^m$,

$$\Gamma^m_{UHF} = \begin{bmatrix} \kappa_1\kappa_2\kappa_3 & \kappa_1\kappa_2(1-\kappa_3) & \kappa_1(1-\kappa_2)\kappa_3 & (1-\kappa_1)\kappa_2\kappa_3 & \kappa_1(1-\kappa_2)(1-\kappa_3) & (1-\kappa_1)\kappa_2(1-\kappa_3) & (1-\kappa_1)(1-\kappa_2)\kappa_3 & (1-\kappa_1)(1-\kappa_2)(1-\kappa_3) \\ 0 & \kappa_1\kappa_2 & 0 & 0 & \kappa_1(1-\kappa_2) & (1-\kappa_1)\kappa_2 & 0 & (1-\kappa_1)(1-\kappa_2) \\ 0 & 0 & \kappa_1\kappa_3 & 0 & \kappa_1(1-\kappa_3) & 0 & (1-\kappa_1)\kappa_3 & (1-\kappa_1)(1-\kappa_3) \\ 0 & 0 & 0 & \kappa_2\kappa_3 & 0 & \kappa_2(1-\kappa_3) & (1-\kappa_2)\kappa_3 & (1-\kappa_2)(1-\kappa_3) \\ 0 & 0 & 0 & 0 & \kappa_1 & 0 & 0 & (1-\kappa_1) \\ 0 & 0 & 0 & 0 & 0 & \kappa_2 & 0 & (1-\kappa_2) \\ 0 & 0 & 0 & 0 & 0 & 0 & \kappa_3 & (1-\kappa_3) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

At any round r, each station i currently in the UHF band has 8 possible states. Let $\Phi_i^r \in \{0, 1\}^8$ be the indicator vector that denotes the current state: if the station is in state S_1, the first binary indicator equals 1, and so on. The probabilities of that station to be in each of the 8 states at round $(r + m)$ are $\Phi_i^r \Gamma^m_{UHF}$, which is a vector of dimension 8 where a component at position $j = 1, \ldots, 8$ represents the probability of being in S_j.

For a station currently in UVH, Figure 2 (b) shows the Markov chain. The states are S_2, S_5, S_6 and S_8. The probability of remaining in S_2 is $q_1 q_2$ because this occurs when the station is willing to accept both the offer to go off-air (with probability q_1) and the offer to be downgraded to LVH (with probability q_2). The probability to go from S_2 to S_5 is $q_1(1 - q_2)$ because this occurs when the station is willing to accept the offer to go off-air and declines the offer to be downgraded to LVH. Similarly, we derive the transition probabilities between other states as shown in the figure. The transition matrix for S_2, S_5, S_6 and S_8 using this ordering is

$$\Gamma_{UVH} = \begin{bmatrix} q_1 q_2 & q_1(1-q_2) & (1-q_1)q_2 & (1-q_1)(1-q_2) \\ 0 & q_1 & 0 & (1-q_1) \\ 0 & 0 & q_2 & (1-q_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is exactly the same as in the transition matrix for Γ_{UHF} between states S_2, S_5, S_6 and S_8. Using the same singular value decomposition technique, we obtain

$$\Gamma^m_{UVH} = \begin{bmatrix} \kappa_1 \kappa_2 & \kappa_1(1-\kappa_2) & (1-\kappa_1)\kappa_2 & (1-\kappa_1)(1-\kappa_2) \\ 0 & \kappa_1 & 0 & (1-\kappa_1) \\ 0 & 0 & \kappa_2 & (1-\kappa_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For a station currently in LVH, Figure 2 (c) shows the Markov chain. The state can be $S_5 = \{\text{OFF}\}$, which means the station is still active to go off-air, or $S_8 = \{\text{None}\}$, which means the station is no longer participating. The transition probabilities are shown in the figure. Here, the transition probability to go from S_5 to S_5 is q_1 while that to go from S_5 to S_8 is $(1-q_1)$. The transition matrix is

$$\Gamma_{LVH} = \begin{bmatrix} q_1 & (1-q_1) \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Gamma^m_{LVH} = \begin{bmatrix} \kappa_1 & (1-\kappa_1) \\ 0 & 1 \end{bmatrix}$$

During the auction, each UVH station will be in state S_2, S_5, S_6, or S_8. We denote by $I^{(r)}_{UVH,i} \in \mathbb{R}^4$ the corresponding probability vector. At the beginning of the auction, $I^{(0)}_{UVH,i} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$. By the end of round m, $I^{(m)}_{UVH,i} = \Gamma_{UVH} I^{(m-1)}_{UVH,i} = \cdots = \Gamma^m_{UVH} I^{(0)}_{UVH,i}$.

Similarly, each LVH station will be in state S_5 or S_8. We denote by $I^{(r)}_{LVH,i} \in \mathbb{R}^2$ the probability vector. At the beginning of the auction, $I^{(0)}_{LVH,i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. By the end of round m, $I^{(m)}_{LVH,i} = \Gamma_{LVH} I^{(m-1)}_{LVH,i} = \cdots = \Gamma^m_{LVH} I^{(0)}_{LVH,i}$. This completes the proof of the theorem.

The significance of Theorem 1 is that, instead of having to keep track of the transitions through m rounds, we can apply the change of variables and view it as a single-round DCA. This helps simplify the expression of the state probabilities, and will help tame the complexity of the nonlinear model as we will show in Section 3.4.

To proceed further, one needs to model how the auctioneer will deal with stations with more than one active option remaining. Which one of those options will the auctioneer offer to the stations? In principle one could treat this as a combinatorial optimization problem with the objective of payment minimization, but that has the same structure as a winner determination problem and is often prohibitively difficult to solve.[5] Instead, we assume that the auctioneer will choose among the bidder's available options with equal probabilities.[6] For example, if a station is in state S_1, that is, it offers to be off-air, downgraded to LVH, or downgraded to UVH, then there is one third probability that it will be taken off-air, one third probability that it will be moved to LVH, and one third probability that it will be moved to UVH.[7]

So, if the auctioneer fixes the percentiles for acceptance to be q_1, q_2, and q_3 for the three options OFF, LVH and UVH, respectively, then by round m,

- the number of stations in state S_1 is $N_{UHF} I^{(m)}_{UHF,i}(1)$,

[5] Solving this WDP is challenging because it involves thousands of binary variables and millions of interference-avoidance constraints. The FCC attempted to solve it when they considered using a seal-bid auction framework for the reverse auction. Solving an instance of this WDP with a state-of-the-art optimization package takes weeks without finding the optimal solution [17].

[6] It is possible to use unequal probabilities as well.

[7] In December 2014 the FCC put out for comment the idea of introducing a 'preferred option' into their DCA; each station that still has multiple active options must state which option is preferred [10]. In Appendix A.1 we discuss how our framework can be extended to capture that and other aspects under consideration by the FCC.

- the number of stations in state S_2 is $N_{UHF}I^{(m)}_{UHF,i}(2) + N_{UVH}I^{(m)}_{UVH,i}(1)$, i.e., it can include stations that came from UHF or UVH,
- the number of stations in state S_3 is $N_{UHF}I^{(m)}_{UHF,i}(3)$,
- the number of stations in state S_4 is $N_{UHF}I^{(m)}_{UHF,i}(4)$,
- the number of stations in state S_5 is $N_{UHF}I^{(m)}_{UHF,i}(5) + N_{UVH}I^{(m)}_{UVH,i}(2) + N_{LVH}I^{(m)}_{LVH,i}(1)$,
- the number of stations in state S_6 is $N_{UHF}I^{(m)}_{UHF,i}(6) + N_{UVH}I^{(m)}_{UVH,i}(3)$,
- the number of stations in state S_7 is $N_{UHF}I^{(m)}_{UHF,i}(7)$, and
- the number of stations in state S_8 is $N_{UHF}I^{(m)}_{UHF,i}(8) + N_{UVH}I^{(m)}_{UVH,i}(4) + N_{LVH}I^{(m)}_{LVH,i}(2)$.

We define the weight matrix $W_{UHF} \in \mathbb{R}^{8\times 4}$ where each row corresponds to a state among 8 possible states, S_1 to S_8, and the four columns correspond to the probabilities of having the final assignment be UHF, UVH, LVH, or OFF, respectively:

$$W_{UHF} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

In this example, the first row states that a station that ended up in state S_1 will have $\frac{1}{3}$ chance to be assigned to each of UVH, LVH, and OFF. The 8th row states that a station that ended up in state S_8, that is, rejected all three options, will be assigned to UHF.

Similarly, the weight matrices for UVH and LVH stations are

$$W_{UVH} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad W_{LVH} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then the expected number of stations that end up in the UHF band is

$$\begin{aligned} E_{UHF} &= \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(1) \\ &= N_{UHF}(S_1)(1-\kappa_1)(1-\kappa_2)(1-\kappa_3) + \\ &\quad N_{UHF}(S_2)(1-\kappa_1)(1-\kappa_2) + N_{UHF}(S_3)(1-\kappa_1)(1-\kappa_3) + N_{UHF}(S_4)(1-\kappa_2)(1-\kappa_3) + \\ &\quad N_{UHF}(S_5)(1-\kappa_1) + N_{UHF}(S_6)(1-\kappa_2) + N_{UHF}(S_7)(1-\kappa_3) + N_{UHF}(S_8), \end{aligned}$$

where $W_{UHF}(j)$ denotes the jth column of W_{UHF}, and $N_{UHF}(S_j)$ is the number of UHF stations currently in state S_j. The expected number of stations that end up in UVH is

$$\begin{aligned} E_{UVH} &= \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(2) + \sum_{i\in UVH} \Phi_i^r \Gamma_{UVH}^m W_{UVH}(1) \\ &= N_{UHF}(S_1)(\kappa_3 - \frac{1}{2}\kappa_1\kappa_3 - \frac{1}{2}\kappa_2\kappa_3 + \frac{1}{3}\kappa_1\kappa_2\kappa_3) + N_{UHF}(S_3)(\kappa_3 - \frac{1}{2}\kappa_1\kappa_3) + \\ &\quad N_{UHF}(S_4)(\kappa_3 - \frac{1}{2}\kappa_2\kappa_3) + N_{UHF}(S_7)(\kappa_3) + N_{UVH}(S_2)(1-\kappa_1)(1-\kappa_2) + \\ &\quad N_{UVH}(S_5)(1-\kappa_1) + N_{UVH}(S_6)(1-\kappa_2) + N_{UVH}(S_8) \end{aligned}$$

Finally, the expected number of stations that end up in the LVH band is

$$\begin{aligned}
E_{LVH} &= \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(3) + \sum_{i\in UVH} \Phi_i^r \Gamma_{UVH}^m W_{UVH}(2) + \sum_{i\in LVH} \Phi_i^r \Gamma_{LVH}^m W_{LVH}(1) \\
&= N_{UHF}(S_1)(\kappa_2 - \frac{1}{2}\kappa_1\kappa_2 - \frac{1}{2}\kappa_2\kappa_3 + \frac{1}{3}\kappa_1\kappa_2\kappa_3) + N_{UHF}(S_2)(\kappa_2 - \frac{1}{2}\kappa_1\kappa_2) + \\
&\quad N_{UHF}(S_4)(\kappa_2 - \frac{1}{2}\kappa_2\kappa_3) + N_{UHF}(S_6)(\kappa_2) + N_{UVH}(S_2)(\kappa_2 - \frac{1}{2}\kappa_1\kappa_2) + \\
&\quad N_{UVH}(S_6)\kappa_2 + N_{LVH}(S_5)(1-\kappa_1) + N_{LVH}(S_8)
\end{aligned}$$

3.3 Percentile-based approach for price setting in each round

A very simple way to adjust prices across DCA rounds is to decrease each price by a given percentage—starting from the upper bound of the support of the station's valuation distribution support for each of the options. We present experiments with that approach later on in the paper.

However, one can do better even within the percentile-based family by using the Markov chain approach presented above. Suppose the auctioneer offers prices to the options so that the acceptance probabilities of the three options are q_1, q_2 and q_3. How should these probabilities be set so that, after m rounds, the expected numbers of stations allocated to the bands match given targets? The auctioneer can solve for (q_1, q_2, q_3) so that the expected number of stations allocated to each band equals its target. For this, we need to find these expectations. This involves finding the probability of the station ending in each band after m rounds. We provide detailed calculation of these expectations in Appendix A.2.

3.4 Optimization model for price setting in each round

We consider the case where the auctioneer wants to offer different acceptance rates to different stations. This will provide more flexibility and hence intuitively should lead to lower aggregate payment by the auctioneer. Instead of aiming to find the same (q_1, q_2, q_3) for all stations, we aim to find the optimal (q_{i1}, q_{i2}, q_{i3}) for each station $i \in \mathcal{N}$.

For notational convenience we denote $d_{ik} = (u_{ik} - l_{ik})$. At round r, the offer price to station i and option k is $p_{ik}^{(1)} = l_{ik} + q_{ik}d_{ik}$.[8] If this option is still available at round $(r+1)$, then the upper bound u_{il} is updated to $p_{ik}^{(1)}$ and the new offer price $p_{ik}^{(2)} = l_{ik} + q_{ik}(p_{ik}^{(1)} - l_{ik}) = l_{ik} + q_{ik}^2 d_{ik}$. Similarly, if option k is still active for station i at round $(r+m)$, then the offer price at that round will be $p_{ik}^{(m)} = l_{ik} + q_{ik}^m d_{ik} = l_{ik} + \kappa_{ik} d_{ik}$.

The payment vector $C_{UHF,i} \in \mathbb{R}^8$ for station i in UHF that corresponds to the 8 possible states is

$$C_{UHF,i} = \begin{bmatrix} \frac{1}{3}(p_{i1}^{(m)} + p_{i2}^{(m)} + p_{i3}^{(m)}) & \frac{1}{2}(p_{i1}^{(m)} + p_{i2}^{(m)}) & \frac{1}{2}(p_{i1}^{(m)} + p_{i3}^{(m)}) & \frac{1}{2}(p_{i2}^{(m)} + p_{i3}^{(m)}) & p_{i1}^{(m)} & p_{i2}^{(m)} & p_{i3}^{(m)} & 0 \end{bmatrix}$$

Similarly, the payment vector $C_{UVH,i} \in \mathbb{R}^4$ for station i in the UVH band that corresponding to 4 possible states is $C_{UVH,i} = \begin{bmatrix} \frac{1}{2}(p_{i1}^{(m)} + p_{i2}^{(m)}) & p_{i1}^{(m)} & p_{i2}^{(m)} & 0 \end{bmatrix}$. The payment vector $C_{LVH,i} \in \mathbb{R}^2$ for station i in the LVH band that corresponding to 2 possible states is $C_{LVH,i} = \begin{bmatrix} p_{i1}^{(m)} & 0 \end{bmatrix}$.

The expected payment at round $(r+m)$ is

$$E[c(\boldsymbol{p})] = \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m C_{UHF,i} + \sum_{i\in UVH} \Phi_i^r \Gamma_{UVH}^m C_{UVH,i} + \sum_{i\in LVH} \Phi_i^r \Gamma_{LVH}^m C_{LVH,i}.$$

[8] The derivations in this section assume that the valuation distributions are uniform. The techniques can be generalized to the nonuniform case, but the final optimization problem can end up having a higher-order polynomial objective and higher-order polynomial constraints than the optimization problem we derive in this section. Also, note that one can use the uniformity assumption in the price setting even if the assumption does not actually hold.

Our OPT-SCHED model for minimizing expected payment while ensuring that the expected number of accepted bidders in each band equals its target is

$$
\begin{array}{ll}
\min_{\boldsymbol{q}} & \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m C_{UHF,i} + \sum_{i\in UVH} \Phi_i^r \Gamma_{UVH}^m C_{UVH,i} + \sum_{i\in LVH} \Phi_i^r \Gamma_{LVH}^m C_{LVH,i}, \\
\text{s.t.} & \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(1) \le C_{UHF} \\
& \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(2) + \sum_{i\in UVH} \Phi_i^r \Gamma_{UVH}^m W_{UVH}(1) \le C_{UVH} \\
& \sum_{i\in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(3) + \sum_{i\in UVH} \Phi_i^r \Gamma_{UVH}^m W_{UVH}(2) + \sum_{i\in LVH} \Phi_i^r \Gamma_{LVH}^m W_{LVH}(1) \le C_{LVH}
\end{array}
\tag{4}
$$

This is a polynomial optimization problem where the objective function is of order $4m$ and the constraint involve polynomials of order $3m$. This is very challenging to solve. We can use the transformation of decision variables from q_{ik} to κ_{ik}. This will lead to a new polynomial optimization problem with the objective function having degree 4 and the constraints having degree 3, which is much more manageable. That fully expanded form is given in Appendix A.3. This problem has $3n$ continuous variables. Solving this problem directly is still not easy due to nonlinearity and nonconvexity. However, the problem has a separable objective and separable constraints. Hence we can apply Lagrangian relaxation. By introducing notation g, h, and u, we can rewrite that problem (i.e., Problem (7)) as

$$
\begin{array}{ll}
\min_{\boldsymbol{\kappa}} & \sum_{i\in\mathcal{N}} f_i(\boldsymbol{\kappa_i}) \\
\text{s.t.} & \sum_{i\in\mathcal{N}} g_i(\boldsymbol{\kappa_i}) \le C_{UHF} \\
& \sum_{i\in\mathcal{N}} h_i(\boldsymbol{\kappa_i}) \le C_{UVH} \\
& \sum_{i\in\mathcal{N}} u_i(\boldsymbol{\kappa_i}) \le C_{LVH}
\end{array}
\tag{5}
$$

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ be the Lagrangian multipliers for the three constraints in Model (5). The Lagrangian dual function is

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{\lambda}, \boldsymbol{\kappa}) &= \sum_{i\in\mathcal{N}} f_i(\boldsymbol{\kappa_i}) + \lambda_1 \left(C_{UHF} - \sum_{i\in\mathcal{N}} g_i(\boldsymbol{\kappa_i}) \right) + \lambda_2 \left(C_{UVH} - \sum_{i\in\mathcal{N}} h_i(\boldsymbol{\kappa_i}) \right) + \lambda_3 \left(C_{LVH} - \sum_{i\in\mathcal{N}} u_i(\boldsymbol{\kappa_i}) \right) \\
&= \lambda_1 C_{UHF} + \lambda_2 C_{UVH} + \lambda_3 C_{LVH} + \sum_{i\in\mathcal{N}} \left(f_i(\boldsymbol{\kappa_i}) \lambda_1 g_i(\boldsymbol{\kappa_i}) + \lambda_2 h_i(\boldsymbol{\kappa_i}) + \lambda_3 u_i(\boldsymbol{\kappa_i}) \right)
\end{aligned}
$$

The Lagrangian dual problem can be derived as

$$
\max_{\lambda\ge 0} \quad \left\{ \lambda_1 C_{UHF} + \lambda_2 C_{UVH} + \lambda_3 C_{LVH} + \sum_{i\in\mathcal{N}} \min_{0\le\boldsymbol{\kappa_i}\le 1} \lambda_1 g_i(\boldsymbol{\kappa_i}) + \lambda_2 h_i(\boldsymbol{\kappa_i}) + \lambda_3 u_i(\boldsymbol{\kappa_i}) \right\}
$$

For each fixed set of Lagrangian multipliers, the Lagrangian dual problem includes n separable nonlinear problems, each with three decision variables lying in the box $[0, 1]^3$ and with an objective that is a 4th-order polynomial that can be solved efficiently. (In the experiments, we used the *Knitro* nonlinear optimization solver to do this.) As the "outer loop" around this subproblem, we apply a conjugate gradient method to solve for the Lagrangian multipliers λ.

4 Finding the best assignment in the final round settlement

After the last round of the auction—i.e., the round where the packing ceases to be feasible—the auctioneer has to decide an outcome for each bidder so as to minimize total payment. For each bidder, the outcome has to be selected from among the options that the bidder's bids indicate are still acceptable to her. In case none of the three options are acceptable, the bidder has to stay in its current band, but can be reallocated to a different channel within the band.

We use the notation from Section 2.1 since the problem of finding the best assignment shares the set of constraints with the feasibility problem. That is, z_{ik} is a binary variable that indicates whether station i is assigned to channel k. We have $\sum_{k\in C_i} z_{ik} \leq 1$ since each station will be assigned to at most one channel. If $\sum_{k\in C_i} z_{ik} = 0$, then the station goes off-air and the auctioneer needs to pay the offer price $p_{i,OFF}$. Otherwise, the station needs to be assigned to one of the three bands. Let $C_{ib} \in C_i$ be the list of feasible channels for station i in band $b \in \{LVH, UVH, UHF\}$. If $\sum_{k\in C_{ib}} z_{ik} = 1$, the auctioneer will pay $p_{i,b}$ to the station to be allocated into band b if that differs from the station's current band. Thus, for an UHF station i, the payment is $p_{i,LVH} \sum_{k\in C_{i,LVH}} z_{ik} + p_{i,UVH} \sum_{k\in C_{i,UVH}} z_{ik} + p_{i,OFF}(1 - \sum_{k\in C_i} z_{ik})$. The payment for a UVH station is $p_{i,LVH} \sum_{k\in C_{i,LVH}} z_{ik} + p_{i,OFF}(1 - \sum_{k\in C_i} z_{ik})$. The total payment for an LVH stations is $p_{i,OFF}(1 - \sum_{k\in C_i} z_{ik})$. Summing the payments to individual stations provides us with the objective function that the FCC wants to minimize. So, the overall final-round winner determination problem is

$$\begin{aligned} \min_{z} \quad & \sum_{i\in\mathcal{N}} p_{i,OFF}(1 - \sum_{k\in C_i} z_{ik}) + \sum_{i\in UVH,UHF} p_{i,LVH}(1 - \sum_{k\in C_{i,LVH}} z_{ik}) + \sum_{i\in UHF} p_{i,UVH}(1 - \sum_{k\in C_{i,UVH}} z_{ik}) \\ \text{s.t.} \quad & \mathbf{z} \in \mathcal{P}(\mathcal{A}, \mathcal{C}); \qquad \text{See (1) for a formulation of } \mathcal{P}(\mathcal{A}, \mathcal{C}). \end{aligned} \tag{6}$$

This is a mixed integer linear program with similar structure to the WDP in the VCG. It has 616,907 binary variables and 2.9×10^6 constraints, which make it very difficult to solve. The FCC has attempted to solve it, but according to Milgrom and Segal [17], solving an instance of it with a state-of-the-art integer programming package takes weeks without finding an optimal solution.

We present a custom mathematical programming technique to find a near-optimal solution. Although the problem has a very large number of decision variables and a huge number of constraints, it has nice underlying structure. Stations can be viewed as nodes and interference constraints as edges. The problem becomes a generalized graph coloring problem with the following additional restrictions. First, each station has a set of feasible channels that it can be allocated to. Second, the adjacency restrictions should be represented by 'dotted edges' in the graph to indicate the certain pairs of stations cannot be allocated to adjacent channels. Observe that the constraints are of knapsack form and many of them can be combined by looking for all the maximal cliques as presented by Nguyen and Sandholm [19]. This reduces the number of constraints and produces a stronger LP relaxation. In addition, with the underlying graph, we apply a decomposition technique to divide the problem into smaller manageable subproblems through Lagrangian relaxation. Although the algorithm may not produce an optimal solution even if we let it run a long time, our experiments show that within a run-time limit of 10 minutes we usually obtain solutions that are 85-90% of optimal. We find this performance acceptable for the purpose of comparing the percentile-based approach and the optimization-based approach. (Without the decomposition, it took CPLEX 10-30 minutes to *load* the problem, not to talk about solving it.) The methods for decomposing the problem and for solving the Lagrangian relaxation are described in Appendix A.4.

5 Experiments

We implemented our optimization-based price-decrementing technique and conducted experiments using real FCC data. We compared the performance against the simple natural percentile-based method (described in the beginning of Section 3.3).

Since no incentive auctions have yet been conducted, we have to use generated data on the bounds of the bidders' valuations. The bounds for the first experiment (symmetric bidders) are generated using a uniform distribution where the upper and lower bound for the off-air option for bidder i are set to $u_{i1} = (1 + \delta)m_i$ and $l_{i1} = (1 - \delta)m_i$ and where m_i is a uniform random variable in $[0, 1]$. Here, $\delta = 0.2$ is a measure of how good the auctioneer's estimate of the bidders' valuations is. For each station, the upper and lower support bounds for the LVH option and the UVH option are set to 66.7% and 33.3% of the bounds for the off-air option, respectively. These percentages are consistent with the FCC estimates FCC [10]. We then draw random sample bid values from these ranges, that is, $\xi_{ik} \sim U[l_{ik}, u_{ik}]$ for each bidder $i = 1, \ldots, n$ and for each option $k = \{1, 2, 3\}$. We draw M = 10 valuation vectors. Each vector corresponds to a DCA instance. The setting for the second experiment (asymmetric bidders) is similar except that the mean value m_i is set proportional to the population that station i serves [7]. In each experiment, the number of rounds allowed is 50.

We tried a number of possible acceptance probabilities for the percentile-based method. Early experiments showed that acceptance probabilities below 0.97 per round often make the DCA auctions end prematurely while acceptance probabilities above 0.995 have little effect on the price discovery. Thus we conducted detailed experiments with acceptance probabilities in $\{0.97, 0.975, 0.98, 0.985, 0.99, 0.995, 1\}$.

We also studied the role of final round settlement. Sections 5.1 and 5.2 assume that once encountering infeasibility, the DCA stops. We then report the final payments that the auctioneer has to pay to all active options of the second-to-last (i.e., last feasible) round. Section 5.3 studies the case with final round settlement.

5.1 Incentive auctions with symmetric valuation distributions

Figures 3 shows that OPT-SCHED outperforms the percentile-based approach on all instances—with 27% reduction in payment compared to the best choice of the per-round acceptance probability in the simple percentile-based approach. The optimization-based approach is able to reject high-value bids taking into account feasibility considerations.

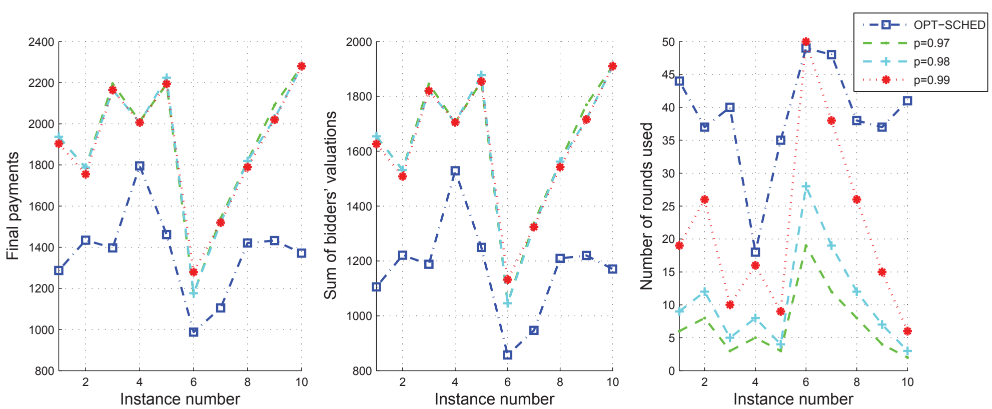


Figure 3: Final payments and active bidders in the symmetric valuations setting.

Figure 4-a illustrates how the auction proceeds across rounds. OPT-SCHED succeeds in proceeding through more rounds before reaching infeasibility by being more intelligent about taking feasibility into account in the pricing. It has a better way of rejecting high-priced bids and to balance payments against feasibility constraints.

5.2 Incentive auctions with asymmetric valuation distributions

In the next experiment, we set the mean value m_i proportional to the population that station i serves. Figure 5 shows the performance over $M = 10$ generated auction instances. OPT-SCHED yields lower final

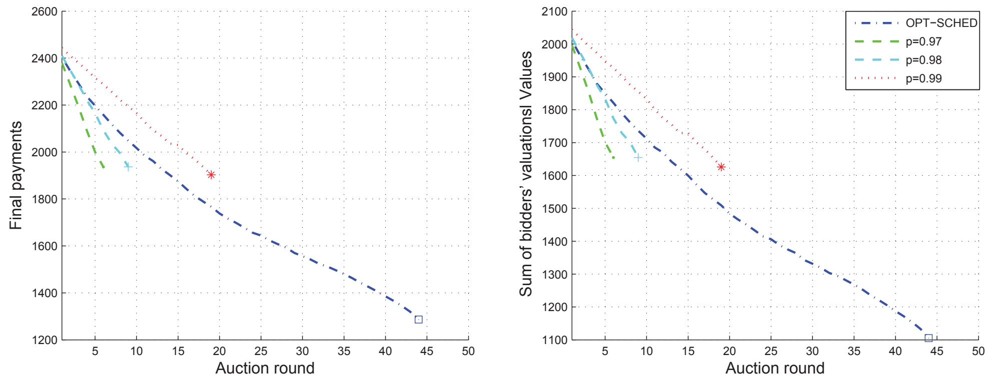


Figure 4: Auction trajectory on Instance 1 (a typical instance) in the symmetric valuations setting.

payment than the simple percentile-based approach for all choices of the fixed acceptance probability. It results in 25% lower payment on average.

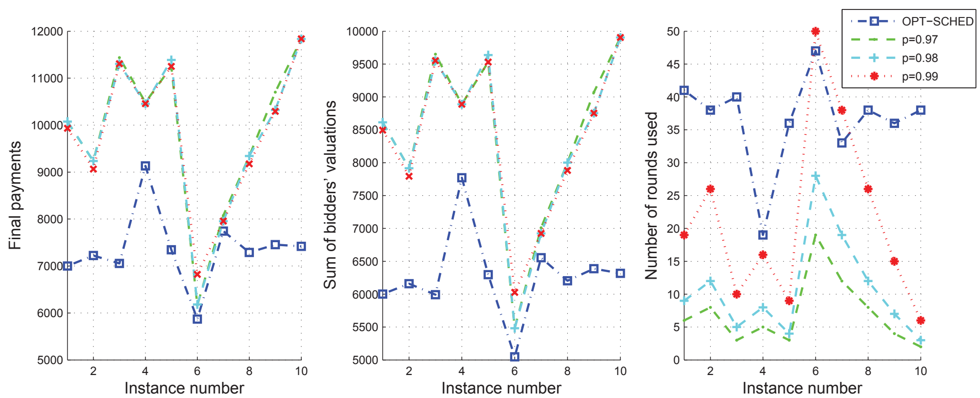


Figure 5: Final payments and active bidders in the asymmetric valuations setting.

Figure 6 shows that, again, OPT-SCHED succeeds in proceeding through more rounds before reaching infeasibility by more intelligently taking feasibility into account in the pricing.

5.3 Effect of final round settlement

The results presented above in Sections 5.1 and 5.2 are measured based on the sum (or equivalently, the average) of the payment to the active options of all bidders just before reaching repacking infeasibility. Instead, the auctioneer can choose the best set of offers from the bidders' active options in order to minimize total payment while ensuring repacking feasibility by solving the final round settlement model from Section 4. Figures 7 and 8 report results with final round settlement. (We do not show the results of the percentile-based approach for some instances because on those instances the payment and sum of bidders' valuations are much larger than on the rest of the instances, and thus their inclusion would make it difficult to visualize the rest of the results. These instances that are particularly bad for the percentile-

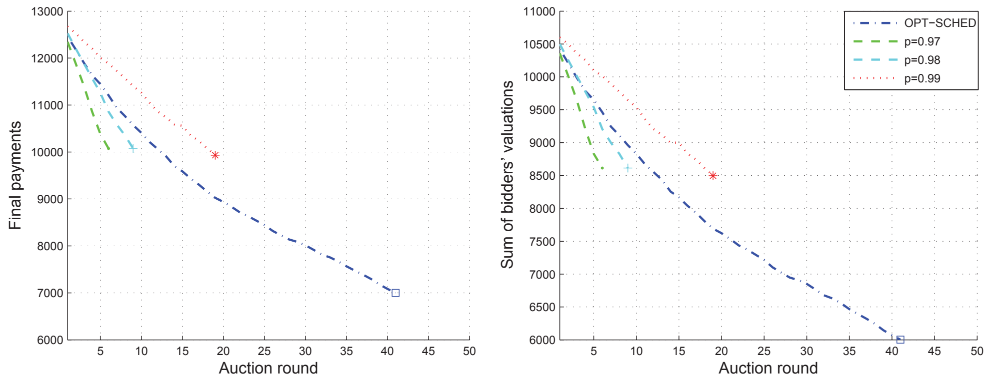


Figure 6: Auction trajectory on Instance 1 (a typical instance) in the asymmetric valuations setting.

based approach include Instance 6 with $p = 0.99$, Instance 7 with $p = 0.97$, and Instance 9 with $p = 0.97$.)

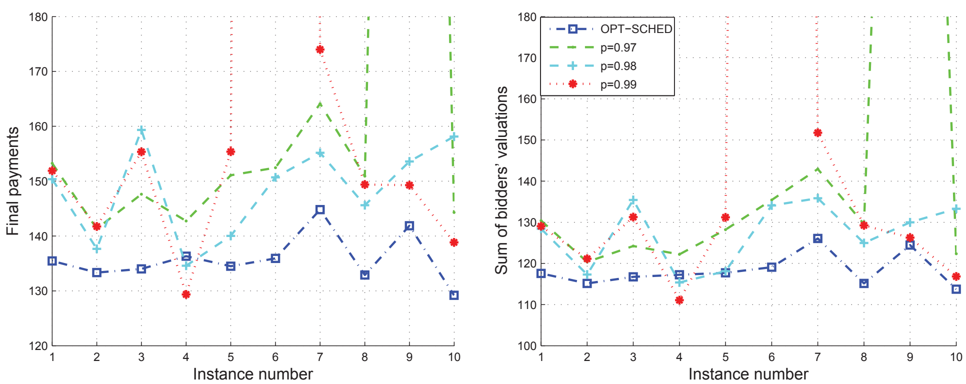


Figure 7: Results with final round settlement in the symmetric valuations setting.

OPT-SCHED outperforms the percentile-based approach for all choices of acceptance probabilities in the asymmetric setting. In the symmetric setting, it almost always outperforms (except on Instance 4 with acceptance probability 0.99). The average payment by OPT-SCHED is 695 while that of the percentile-base approach is 1182, 823, and 1450 for the per-round acceptance probabilities 0.97, 0.98, and 0.99, respectively. In summary, OPT-SCHED dramatically outperforms the percentile-based approach.

5.3.1 Repacking solution

We report the OPT-SCHED repacking solution for Instance 1. The pattern on the other nine instances was similar.

In the asymmetric setting, among the 1647 UHF stations, 1278 were repacked to a UHF channel (channels 14-31), 57 were repacked to a UVH channel (7-13), 65 were repacked to an LVH channel (2-6), and 247 went off-air. Among the 428 UVH stations, 346 were repacked to a UVH channel, 21 were repacked

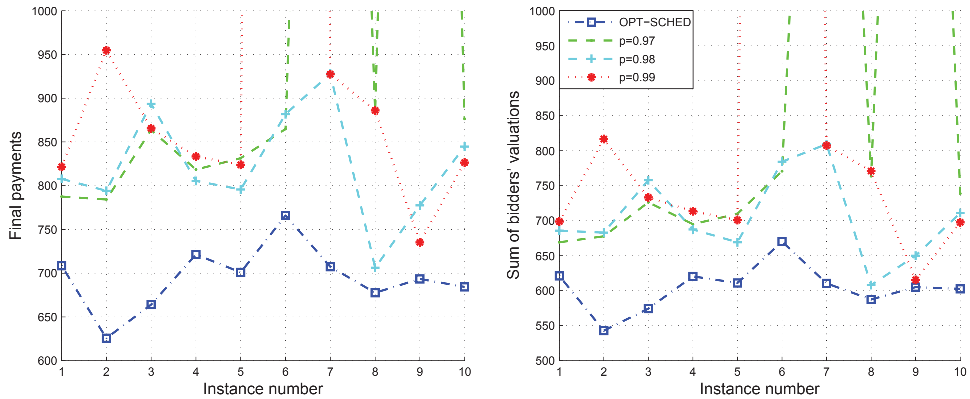


Figure 8: Results with final round settlement in the asymmetric valuations setting.

to an LVH channel, and 61 went off-air. Among the 55 LVH stations, 49 were repacked to an LVH channel and 6 went off-air.

In the symmetric setting, among the 1647 UHF stations, 1277 were repacked to a UHF channel, 54 were repacked to a UVH channel, 56 were repacked to an LVH channel, and 260 went off-air. Among the 428 UVH stations, 341 were repacked to a UVH channel, 20 were repacked to an LVH channel, and 67 went off-air. Among the 55 LVH stations, 53 were repacked to a LVH channel and 2 went off-air.

6 Conclusions

We presented a multi-option DCA framework in which each bidder may be able to sell one from a set of options to the auctioneer. We developed a Markov chain model for representing the dynamics of each bidder's state in the auction, as well as an optimization model and technique for finding prices to offer to the different bidders for the different options in each round—using the Markov chain. The optimization minimizes total payment while ensuring feasibility in a stochastic sense. We also introduced percentile-based approaches to decrementing prices. Experiments with real FCC incentive auction interference constraint data revealed that the optimization-based approach dramatically outperforms the simple percentile-based approach both under symmetric and asymmetric bidder valuation distributions—because it takes feasibility into account in pricing. Both of our pricing techniques scale to the large.

Bibliography

[1] Adomavicius, G., Curley, S. P., Gupta, A., and Sanyal, P. 2012. Effect of information feedback on bidder behavior in continuous combinatorial auctions. *Mgmt. Sci. 58,* 4, 811–830.

[2] Cramton, P. 1997. The FCC spectrum auctions: an early assessment. *Journal of Economics and Management Strategy 6,* 3, 431–495. Special issue on market design and spectrum auctions.

[3] Cramton, P., Shoham, Y., and Steinberg, R. 2006. *Combinatorial Auctions.* MIT Press.

[4] Day, R. and Milgrom, P. 2008. Core-selecting package auctions. *International Journal of Game Theory 36,* 3, 393–407.

[5] Day, R. W. and Raghavan, S. 2007. Fair payments for efficient allocations in public sector combinatorial auctions. *Management Science 53,* 9, 1389–1406.

[6] FCC. 2012. Notice of proposed rulemaking. Tech. rep., FCC. http://hraunfoss.fcc.gov/edocs_public/attachmatch/FCC-12-118A1.pdf.

[7] FCC. 2013. Fcc baseline data and maps. Tech. rep., FCC. http://data.fcc.gov/download/incentive-auctions/OET-69/Baseline_Data_and_Maps_2013July.pdf.

[8] FCC. 2014a. Broadcast incentive auction 101. Tech. rep., FCC. http:// wireless.fcc.gov/ incentiveauctions/learn-program/Broadcast_Incentive_Auction_101_slides.pdf.

[9] FCC. 2014b. FCC repacking constraint files. http://data.fcc.gov/download/incentive-auctions/Constraint_Files/.

[10] FCC. 2014c. Notice of proposed rulemaking. Tech. rep., FCC. https://apps.fcc.gov/edocs_public/attachmatch/FCC-14-191A1.pdf.

[11] FCC. 2014d. The path to a successful incentive auction. FCC presentation 1/30/2014, wireless.fcc.gov/incentiveauctions/learn-program/Incentive_Auction_Jan_30_Present_9am.pdf.

[12] Hazlett, T., Porter, D., and Smith, V. 2012. "Incentive auctions" economic and strategic issues. Arlington Economics white paper 6/12/2012.

[13] Kelly, F. and Steinberg, R. 2000. A combinatorial auction with multiple winners for universal service. *Management Science 46,* 4, 586–596.

[14] Leyton-Brown, K. 2013. Investigating the viability of exact feasibility testing. Working Paper, University of British Columbia / Auctionomics, http://www.cs.ubc.ca/ ˜ kevinlb/talk.php?u=2013-Feasibility-Testing.pdf.

[15] Milgrom, P. 2004. *Putting Auction Theory to Work*. Cambridge University Press.

[16] Milgrom, P. and Segal, I. 2012. Heuristic auctions and U.S. spectrum repurposing. Presentation, Stanford University, www.stanford.edu/ ˜ isegal/fcc.ppt.

[17] Milgrom, P. and Segal, I. 2013. Deferred-acceptance heuristic auctions. Working Paper, Stanford University, http://www.milgrom.net/downloads/heuristic.pdf.

[18] Milgrom, P. and Segal, I. 2014. Deferred-acceptance auctions and radio spectrum reallocation. In *Proc. ACM Conference on Economics and Computation (EC).*

[19] Nguyen, T.-D. and Sandholm, T. 2014. Optimizing prices in descending clock auctions. In *Proc. ACM Conference on Economics and Computation (EC)*. 93–110.

[20] Olivares, M., Weintraub, G. Y., Epstein, R., and Yung, D. Combinatorial auctions for procurement: An empirical study of the chilean school meals auction. *Mgmt. Sci. 58,* 8.

[21] Paul Duetting, V. G. and Roughgarden, T. 2014. The performance of deferred-acceptance auctions. In *Proc. ACM Conference on Economics and Computation (EC).*

[22] Rothkopf, M. H., Pekeč, A., and Harstad, R. M. 1998. Computationally manageable combinational auctions. *Management science 44,* 8, 1131–1147.

[23] Sandholm, T. 2002. Algorithm for optimal winner determination in combinatorial auctions. *Artificial Intelligence 135*, 1–54.

[24] Sandholm, T. 2006. Optimal winner determination algorithms. In *Combinatorial Auctions*, P. Cramton, Y. Shoham, and R. Steinberg, Eds. MIT Press, 337–368. Chapter 14.

[25] Sandholm, T. 2013. Very-large-scale generalized combinatorial multi-attribute auctions: Lessons from conducting $60 billion of sourcing. In *Handbook of Market Design*, Z. Neeman, A. Roth, and N. Vulkan, Eds. Oxford University Press.

[26] Sandholm, T., Suri, S., Gilpin, A., and Levine, D. 2005. CABOB: A fast optimal algorithm for winner determination in combinatorial auctions. *Management Science 51,* 3, 374–390.

[27] Wheeler, T. 2013. The path to a successful incentive auction. FCC Chairman's post on the official FCC blog 12/6/2013.

APPENDIX

A.1 Incorporating a one-way hierarchy of options

In December 2014, the FCC put up for comment a proposal for the DCA to be used for the reverse auction part of the imminent incentive auction, including a sketch of a price adjustment method [10]. It is much more rigid than what we propose in this paper. Also, the pricing heuristic does not take feasibility into account to nearly the same extent as our pricing technique does. To our knowledge, no theory or experiments have been published so far to analyze the FCC-proposed design choices.

There is also another confining—but potentially interesting—aspect of that DCA design. The options are considered to form a hierarchy. A bidder has to declare a preferred option (which is the option that he might get) at each point in the auction. A bidder is allowed to move the declared preferred option only downward in the hierarchy. So, a bidder can go from off-air to a lower band to an even lower band and then to accepting no offer, but not in the other direction. Also, a bidder is allowed to only move downward in the hierarchy from the option that she holds before the auction begins. Our Markov modeling and optimization techniques can be adapted to that setting as well. In the rest of this appendix we describe how to do that.

We denote by $k = \{1, 2, 3\}$ the options that correspond to Off-air, LVH and UVH. We denote by p_{ik} the price the auction offers to station i for option k. For simplicity, here we assume the v_{ik} are uniformly distributed with support $[l_{ik}, u_{ik}]$.

At the beginning of each DCA round, the station receives offer prices for its active options and evaluates its surplus $(p_{ik} - v_{ik})$ for each option k. Whenever the station switches its preferred option, options higher in the hierarchy become permanently inactive.

Figure 9 shows the states and transition probabilities for a station that is currently in the UHF band. Each node represents a state and the words inside describe the options that are still active. One of these active options is written in bold and underlined to highlight it as the preferred option. This preferred option is always the highest of the remaining active ones in the hierarchy. There are eight states which include:

- State S_1 with all three options still active and with off-air being the preferred option.
- State S_2 with options off-air and LVH still active and with off-air being the preferred option.
- State S_3 with options off-air and UVH still active and with off-air being the preferred option.
- State S_4 with options LVH and UVH still active and with LVH being the preferred option.
- State S_5 with the only option off-air still active which is also the preferred option.
- State S_6 with the only option LVH still active which is also the preferred option.
- State S_7 with the only option UVH still active which is also the preferred option.

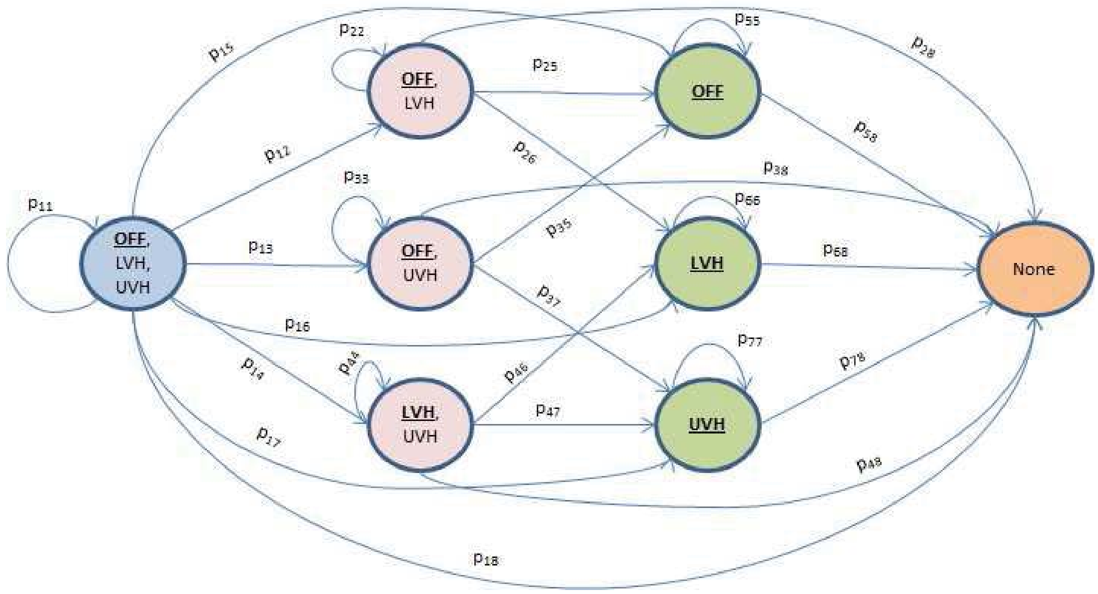


Figure 9: Markov chain on bidder status.

- State S_8 with none of the options active.

We observe that the four states of S_1, S_2,S_3 and S_5 still have off-air as an active and the preferred option. The transition among these four states requires at least two conditions: (a) off-air is still active and (b) off-air is still the best option among those available. The remaining four states of S_4, S_6, S_7 and S_8 do not include the off-air option. This mean that transition from states (S_1, S_2, S_3, S_5) to the four states of (S_4, S_6, S_7, S_8) requires at least either (a) off-air no longer being active or (b) off-air still being acceptable but not as attractive as some other options and hence the station requested a switch. In the latter case, the switch would require delisting the off-air option and going down the hierarchy.

Due to the inclusion of the preferred option which could change throughout the auction, the transition among states is now different. We use the same notation of q_1, q_2, and q_3 being the acceptance probabilities to offer to the three options of off-air, LVH and UVH. We aim to derive the transition probabilities as a function of (q_1, q_2, q_3). Let us denote by p_{ij} the transition probability from state S_i to state S_j.

First let us consider the transition from state S_1 to S_2. This occurs under the following conditions: the UVH option is no longer attractive while the off-air and LVH options are still acceptable with off-air being the preferred option. This transition probability is calculated as

$$\begin{aligned} p_{12} &= \text{Prob}(p_{i1} - v_{i1} \geq p_{i2} - v_{i2} \geq 0 \geq p_{i3} - v_{i3}) \\ &= \text{Prob}(p_{i1} - v_{i1} \geq p_{i2} - v_{i2} \geq 0)\text{Prob}(p_{i3} - v_{i3} \leq 0) \end{aligned}$$

We observe that $(p_{ik} - v_{ik})$ is a uniform random variable with support $[p_{ik} - u_{ik}, p_{ik} - l_{ik}]$. Thus,

$\text{Prob}(p_{i3} - v_{i3} \le 0) = \frac{u_{i3}-p_{i3}}{u_{i3}-l_{i3}}$. In addition,

$$\begin{aligned}
\text{Prob}(p_{i1} - v_{i1} \ge p_{i2} - v_{i2} \ge 0) &= \int_0^{min(p_{i1}-l_{i1}, p_{i2}-l_{i2})} \left[\int_x^{p_{i1}-l_{i1}} \frac{1}{d_1} dy\right] \frac{1}{d_2} dx \\
&= \begin{cases} \frac{1}{d_1 d_2} \int_0^{p_{i2}-l_{i2}} [p_{i1} - l_{i1} - x]\, dx, & \text{if } p_{i2} - l_{i2} \le p_{i1} - l_{i1} \\ \frac{1}{d_1 d_2} \int_0^{p_{i1}-l_{i1}} [p_{i1} - l_{i1} - x]\, dx, & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{d_1 d_2} \frac{(p_{i2}-l_{i2})(2(p_{i1}-l_{i1})-(p_{i2}-l_{i2}))}{2}, & \text{if } p_{i2} - l_{i2} \le p_{i1} - l_{i1} \\ \frac{1}{d_1 d_2} \frac{(p_{i1}-l_{i1})^2}{2}, & \text{otherwise} \end{cases}
\end{aligned}$$

Thus,

$$p_{12} = \begin{cases} \frac{u_{i3}-p_{i3}}{u_{i3}-l_{i3}} \frac{1}{d_1 d_2} \frac{(p_{i2}-l_{i2})(2(p_{i1}-l_{i1})-(p_{i2}-l_{i2}))}{2}, & \text{if } p_{i2} - l_{i2} \le p_{i1} - l_{i1} \\ \frac{u_{i3}-p_{i3}}{u_{i3}-l_{i3}} \frac{1}{d_1 d_2} \frac{(p_{i1}-l_{i1})^2}{2}, & \text{otherwise} \end{cases}$$

Similarly, we can derive

$$p_{13} = \begin{cases} \frac{p_{i2}-l_{i2}}{u_{i2}-l_{i2}} \frac{1}{d_1 d_3} \frac{(p_{i3}-l_{i3})(2(p_{i1}-l_{i1})-(p_{i3}-l_{i3}))}{2}, & \text{if } p_{i3} - l_{i3} \le p_{i1} - l_{i1} \\ \frac{p_{i2}-l_{i2}}{u_{i2}-l_{i2}} \frac{1}{d_1 d_3} \frac{(p_{i1}-l_{i1})^2}{2}, & \text{otherwise} \end{cases}$$

The transition from state S_1 to S_4 occurs when off-air is not as attractive as LVH while both LVH and UVH are acceptable, that is,

$$\begin{aligned}
p_{14} &= \text{Prob}(p_{i1} - v_{i1} < p_{i2} - v_{i2} \text{ and } 0 \le p_{i2} - v_{i2} \text{ and } 0 \le p_{i3} - v_{i3}) \\
&= \text{Prob}(p_{i3} - v_{i3} \ge 0)\text{Prob}(p_{i1} - v_{i1} < p_{i2} - v_{i2} \text{ and } 0 \le p_{i2} - v_{i2}) \\
&= \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}} \text{Prob}(p_{i1} - v_{i1} < p_{i2} - v_{i2} \text{ and } 0 \le p_{i2} - v_{i2}) \\
&= \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}} \int_0^{p_{i2}-l_{i2}} \left[\int_{p_{i1}-u_{i1}}^{min(x, p_{i1}-l_{i1})} \frac{1}{d_1} dy\right] \frac{1}{d_2} dx \\
&= \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}} \frac{1}{d_1 d_2} \int_0^{p_{i2}-l_{i2}} [min(x, p_{i1} - l_{i1}) - (p_{i1} - u_{i1})]\, dx \\
&= \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}} \frac{1}{d_1 d_2} \begin{cases} \frac{(p_{i2}-l_{i2})(2(u_{i1}-p_{i1})+(p_{i2}-l_{i2}))}{2}, & \text{if } p_{i2} - l_{i2} \le p_{i1} - l_{i1} \\ d_1((p_{i2} - l_{i2}) - (p_{i1} - l_{i1})) + \frac{(p_{i1}-l_{i1})(2(u_{i1}-p_{i1})+(p_{i1}-l_{i1}))}{2}, & \text{otherwise} \end{cases}
\end{aligned}$$

Similarly, we can derive

$$\begin{aligned}
p_{15} &= \text{Prob}(p_{i1} - v_{i1} \ge 0 \text{ and } p_{i2} - v_{i2} \le 0 \text{ and } p_{i3} - v_{i3} \le 0) \\
&= \frac{p_{i1} - l_{i1}}{u_{i1} - l_{i1}} \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}}
\end{aligned}$$

$$\begin{aligned}
p_{16} &= \text{Prob}(p_{i1} - v_{i1} \le 0 \text{ and } p_{i2} - v_{i2} \ge 0 \text{ and } p_{i3} - v_{i3} \le 0) \\
&= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \frac{p_{i2} - l_{i2}}{u_{i2} - l_{i2}} \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}}
\end{aligned}$$

$$\begin{aligned}
p_{17} &= \text{Prob}(p_{i1} - v_{i1} \le 0 \text{ and } p_{i2} - v_{i2} \le 0 \text{ and } p_{i3} - v_{i3} \ge 0) \\
&= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}}
\end{aligned}$$

$$\begin{aligned} p_{18} &= \text{Prob}(p_{i1} - v_{i1} \leq 0 \text{ and } p_{i2} - v_{i2} \leq 0 \text{ and } p_{i3} - v_{i3} \leq 0) \\ &= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}} \end{aligned}$$

From state S_2:

$$\begin{aligned} p_{25} &= \text{Prob}(p_{i1} - v_{i1} \geq 0 \text{ and } p_{i2} - v_{i2} \leq 0) \\ &= \frac{p_{i1} - l_{i1}}{u_{i1} - l_{i1}} \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \end{aligned}$$

$$\begin{aligned} p_{26} &= \text{Prob}(p_{i1} - v_{i1} \leq 0 \text{ and } p_{i2} - v_{i2} \geq 0) \\ &= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \frac{p_{i2} - l_{i2}}{u_{i2} - l_{i2}} \end{aligned}$$

$$\begin{aligned} p_{28} &= \text{Prob}(p_{i1} - v_{i1} \leq 0 \text{ and } p_{i2} - v_{i2} \leq 0) \\ &= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \end{aligned}$$

We can derive p_{22} in a similar way or we can simply use the formulation $p_{22} = 1 - p_{25} - p_{26} - p_{28}$.
From state S_3:

$$\begin{aligned} p_{35} &= \text{Prob}(p_{i1} - v_{i1} \geq 0 \text{ and } p_{i3} - v_{i3} \leq 0) \\ &= \frac{p_{i1} - l_{i1}}{u_{i1} - l_{i1}} \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}} \end{aligned}$$

$$\begin{aligned} p_{37} &= \text{Prob}(p_{i1} - v_{i1} \leq 0 \text{ and } p_{i3} - v_{i3} \geq 0) \\ &= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}} \end{aligned}$$

$$\begin{aligned} p_{38} &= \text{Prob}(p_{i1} - v_{i1} \leq 0 \text{ and } p_{i3} - v_{i3} \leq 0) \\ &= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}} \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}} \end{aligned}$$

We have $p_{33} = 1 - p_{35} - p_{37} - p_{38}$.

$$\begin{aligned} p_{46} &= \text{Prob}(p_{i2} - v_{i2} \geq 0 \text{ and } p_{i3} - v_{i3} \leq 0) \\ &= \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}} \end{aligned}$$

$$\begin{aligned} p_{47} &= \text{Prob}(p_{i2} - v_{i2} \leq 0 \text{ and } p_{i3} - v_{i3} \geq 0) \\ &= \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}} \end{aligned}$$

$$
\begin{aligned}
p_{48} &= \text{Prob}(p_{i2} - v_{i2} \leq 0 \text{ and } p_{i3} - v_{i3} \leq 0) \\
&= \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}} \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}}
\end{aligned}
$$

We have $p_{44} = 1 - p_{46} - p_{47} - p_{48}$.
From state S_5:

$$
\begin{aligned}
p_{58} &= \text{Prob}(p_{i1} - v_{i1} \leq 0) \\
&= \frac{u_{i1} - p_{i1}}{u_{i1} - l_{i1}}
\end{aligned}
$$

$$
\begin{aligned}
p_{55} &= \text{Prob}(p_{i1} - v_{i1} \geq 0) \\
&= \frac{p_{i1} - l_{i1}}{u_{i1} - l_{i1}}
\end{aligned}
$$

From state S_6:

$$
\begin{aligned}
p_{68} &= \text{Prob}(p_{i2} - v_{i2} \leq 0) \\
&= \frac{u_{i2} - p_{i2}}{u_{i2} - l_{i2}}
\end{aligned}
$$

$$
\begin{aligned}
p_{66} &= \text{Prob}(p_{i2} - v_{i2} \geq 0) \\
&= \frac{p_{i2} - l_{i2}}{u_{i2} - l_{i2}}
\end{aligned}
$$

From state S_7:

$$
\begin{aligned}
p_{78} &= \text{Prob}(p_{i3} - v_{i3} \leq 0) \\
&= \frac{u_{i3} - p_{i3}}{u_{i3} - l_{i3}}
\end{aligned}
$$

$$
\begin{aligned}
p_{77} &= \text{Prob}(p_{i3} - v_{i3} \geq 0) \\
&= \frac{p_{i3} - l_{i3}}{u_{i3} - l_{i3}}
\end{aligned}
$$

Figure 10 shows the state transition for stations that are currently in the LVH and UVH bands. For UVH stations, the Markov chain contains four states S_2, S_5, S_6 and S_8. The transition probabilities among these states are equal to those in the Markov chain for an UHF station. Similarly the Markov chain for an LVH station contains two states S_7 and S_8, also with the same transition probabilities

One disadvantage of this new setting with a preferred option and a hierarchy is that the resulting optimization model will be of more complicated form as a result of the new transition probabilities. We leave the computational approach for future research. However, a nice property of the new setting is that the auctioneer pays each station the price of the preferred option and hence we do not have to introduce the weight matrices presented in Section 3.2. In addition, since we know which band each station will be allocated to, the expected number of stations allocated to each band can be calculated with certainty.

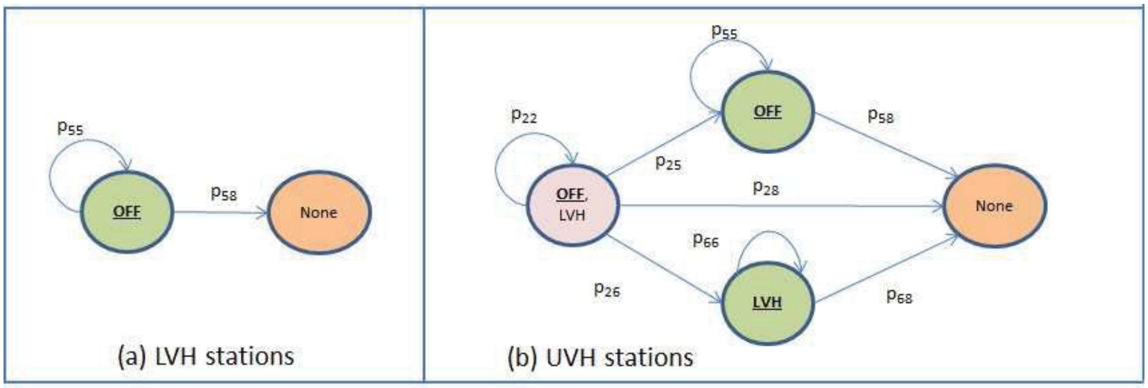


Figure 10: Markov chain on bidder status.

A.2 Method for optimizing percentiles

Recall that for the purposes of this appendix, the auctioneer is trying to solve for (q_1, q_2, q_3) such that the expected number of stations allocated to each band (UHF, UVH, and LVH) equals its target. Denote these targets by $C_b,\ b \in \{\text{UHF, UVH,LVH}\}$.

We get three equations that correspond to three bands and three unknowns. The issue is that these expected values are polynomial of degree $3m$ in the unknown, and numerical methods for solving these might lead to approximation errors. We resolve this challenge by observing that the decision variables (q_1, q_2, q_3) can be replaced by $(\kappa_1, \kappa_2, \kappa_3)$ and we arrive at the following system of three equations for three unknowns $(\kappa_1, \kappa_2, \kappa_3)$:

$$\begin{aligned}
C_{UHF} &= N_{UHF}(S_1)(1-\kappa_1)(1-\kappa_2)(1-\kappa_3) + \\
&\quad N_{UHF}(S_2)(1-\kappa_1)(1-\kappa_2) + N_{UHF}(S_3)(1-\kappa_1)(1-\kappa_3) + N_{UHF}(S_4)(1-\kappa_2)(1-\kappa_3) + \\
&\quad N_{UHF}(S_5)(1-\kappa_1) + N_{UHF}(S_6)(1-\kappa_2) + N_{UHF}(S_7)(1-\kappa_3) + N_{UHF}(S_8) \\
C_{UVH} &= N_{UHF}(S_1)(\kappa_3 - \frac{1}{2}\kappa_1\kappa_3 - \frac{1}{2}\kappa_2\kappa_3 + \frac{1}{3}\kappa_1\kappa_2\kappa_3) + \\
&\quad N_{UHF}(S_3)(\kappa_3 - \frac{1}{2}\kappa_1\kappa_3) + N_{UHF}(S_4)(\kappa_3 - \frac{1}{2}\kappa_2\kappa_3) + N_{UHF}(S_7)(\kappa_3) + \\
&\quad N_{UVH}(S_2)(1-\kappa_1)(1-\kappa_2) + N_{UVH}(S_5)(1-\kappa_1) + N_{UVH}(S_6)(1-\kappa_2) + N_{UVH}(S_8) \\
C_{LVH} &= N_{UHF}(S_1)(\kappa_2 - \frac{1}{2}\kappa_1\kappa_2 - \frac{1}{2}\kappa_2\kappa_3 + \frac{1}{3}\kappa_1\kappa_2\kappa_3) + \\
&\quad N_{UHF}(S_2)(\kappa_2 - \frac{1}{2}\kappa_1\kappa_2) + N_{UHF}(S_4)(\kappa_2 - \frac{1}{2}\kappa_2\kappa_3) + N_{UHF}(S_6)(\kappa_2) + \\
&\quad N_{UVH}(S_2)(\kappa_2 - \frac{1}{2}\kappa_1\kappa_2) + N_{UVH}(S_6)\kappa_2 + N_{LVH}(S_5)(1-\kappa_1) + N_{LVH}(S_8).
\end{aligned}$$

We observe that the right-hand sides of these equalities include polynomials of degree 3. Hence we can apply a numerical method—such as the Newton-Raphson's method—to solve for $(\kappa_1, \kappa_2, \kappa_3)$ easily. We then set $q_i = \kappa_i^{1/m}, \forall i \in \{1, 2, 3\}$.

In the case where the auctioneer sets the same acceptance rates for the three options, let $q = q_1 = q_2 = q_3$

and $r = q^m$. Then the expected numbers of stations in the bands are

$$\begin{aligned}
E_{UHF} &= \sum_{i \in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(1) \\
&= N_{UHF}(S_1)(1-r)^3 + N_{UHF}(S_2,S_3,S_4)(1-r)^2 + N_{UHF}(S_5,S_6,S_7)(1-r) + N_{UHF}(S_8) \\
E_{UVH} &= \sum_{i \in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(2) + \sum_{i \in UVH} \Phi_i^r \Gamma_{UVH}^m W_{UVH}(1) \\
&= N_{UHF}(S_1)(r - r^2 + \frac{1}{3}r^3) + N_{UHF}(S_3,S_4)(r - \frac{1}{2}r^2) + N_{UHF}(S_7)r + \\
&\quad N_{UVH}(S_2)(1-r)^2 + N_{UVH}(S_5,S_6)(1-r) + N_{UVH}(S_8) \\
E_{LVH} &= \sum_{i \in UHF} \Phi_i^r \Gamma_{UHF}^m W_{UHF}(3) + \sum_{i \in UVH} \Phi_i^r \Gamma_{UVH}^m W_{UVH}(2) + \sum_{i \in LVH} \Phi_i^r \Gamma_{LVH}^m W_{LVH}(1) \\
&= N_{UHF}(S_1)(r - r^2 + \frac{1}{3}r^3) + N_{UHF}(S_2,S_4)(r - \frac{1}{2}r^2) + N_{UHF}(S_6)r + \\
&\quad N_{UVH}(S_2)(3/2r - r^2) + N_{UVH}(S_6)r + N_{LVH}(S_5)(1-r) + N_{LVH}(S_8)
\end{aligned}$$

We can then solve for r that minimizes the sum of square errors, i.e., $\sum_b (E_b - C_b)^2$.

A.3 Expanded form of the OPT-SCHED model

Let $UHF(S_j)$ (and similarly $UVH(S_j)$, $LVH(S_j)$) denote the list of UHF (UVH, LVH) stations that are currently in state S_j. The problem of minimizing the expected payment while ensuring the expected number of stations allocated to each band does not exceed the band capacity can be formulated as

$$\begin{aligned}
\min_{\boldsymbol{\kappa}} \quad & f(\boldsymbol{\kappa}) \\
\text{s.t.} \quad & \sum_{i \in UHF(S_1)} (1-\kappa_{i1})(1-\kappa_{i2})(1-\kappa_{i3}) + \\
& \sum_{i \in UHF(S_2)} (1-\kappa_{i1})(1-\kappa_{i2}) + \sum_{i \in UHF(S_3)} (1-\kappa_{i1})(1-\kappa_{i3}) + \sum_{i \in UHF(S_4)} (1-\kappa_{i2})(1-\kappa_{i3}) + \\
& \sum_{i \in UHF(S_5)} (1-\kappa_{i1}) + \sum_{i \in UHF(S_6)} (1-\kappa_{i2}) + \sum_{i \in UHF(S_7)} (1-\kappa_{i3}) + \sum_{i \in UHF(S_8)} 1 \le C_{UHF}, \\
& \sum_{i \in UHF(S_1)} (\kappa_{i3} - \frac{1}{2}\kappa_{i1}\kappa_{i3} - \frac{1}{2}\kappa_{i2}\kappa_{i3} + \frac{1}{3}\kappa_{i1}\kappa_{i2}\kappa_{i3}) + \\
& \sum_{i \in UHF(S_3)} (\kappa_{i3} - \frac{1}{2}\kappa_{i1}\kappa_{i3}) + \sum_{i \in UHF(S_4)} (\kappa_{i3} - \frac{1}{2}\kappa_{i2}\kappa_{i3}) + \sum_{i \in UHF(S_7)} (\kappa_{i3}) + \\
& \sum_{i \in UVH(S_2)} (1-\kappa_{i1})(1-\kappa_{i2}) + \sum_{i \in UVH(S_5)} (1-\kappa_{i1}) + \sum_{i \in UVH(S_6)} (1-\kappa_{i2}) + \sum_{i \in UVH(S_8)} 1 \le C_{UVH}, \\
& \sum_{i \in UHF(S_1)} (\kappa_{i2} - \frac{1}{2}\kappa_{i1}\kappa_{i2} - \frac{1}{2}\kappa_{i2}\kappa_{i3} + \frac{1}{3}\kappa_{i1}\kappa_{i2}\kappa_{i3}) + \\
& \sum_{i \in UHF(S_2)} (\kappa_{i2} - \frac{1}{2}\kappa_{i1}\kappa_{i2}) + \sum_{i \in UHF(S_4)} (\kappa_{i2} - \frac{1}{2}\kappa_{i2}\kappa_{i3}) + \sum_{i \in UHF(S_6)} (\kappa_{i2}) + \\
& \sum_{i \in UVH(S_2)} (\kappa_{i2} - \frac{1}{2}\kappa_{i1}\kappa_{i2}) + \sum_{i \in UVH(S_5)} \kappa_{i2} + \sum_{i \in LVH, \Phi_i = S_6} (1-\kappa_{i1}) + \sum_{i \in LVH, \Phi_i = S_8} 1 \le C_{LVH},
\end{aligned} \tag{7}$$

where

$$
\begin{aligned}
f(\boldsymbol{\kappa}) &= \sum_{i\in UHF}\Phi_i^r\Gamma_{UHF}^m C_{UHF,i} + \sum_{i\in UVH}\Phi_i^r\Gamma_{UVH}^m C_{UVH,i} + \sum_{i\in LVH}\Phi_i^r\Gamma_{LVH}^m C_{LVH,i}, \\
&= \sum_{i\in UHF(S_1)}\frac{1}{3}\kappa_{i1}\kappa_{i2}\kappa_{i3}(\sum_{k=1}^{3} l_{ik}+\kappa_{ik}d_{ik}) + \frac{1}{2}\kappa_{i1}\kappa_{i2}(1-\kappa_{i3})(\sum_{k=1,2} l_{ik}+\kappa_{ik}d_{ik}) + \\
&\quad \frac{1}{2}\kappa_{i1}(1-\kappa_{i2})\kappa_{i3}(\sum_{k=1,3} l_{ik}+\kappa_{ik}d_{ik}) + \frac{1}{2}(1-\kappa_{i1})\kappa_{i2}\kappa_{i3}(\sum_{k=2,3} l_{ik}+\kappa_{ik}d_{ik}) + \\
&\quad \kappa_{i1}(1-\kappa_{i2})(1-\kappa_{i3})(l_{i1}+\kappa_{i1}d_{i1}) + (1-\kappa_{i1})\kappa_{i2}(1-\kappa_{i3})(l_{i2}+\kappa_{i2}d_{i2}) + \\
&\quad (1-\kappa_{i1})(1-\kappa_{i2})\kappa_{i3}(l_{i3}+\kappa_{i3}d_{i3}) + \\
&\quad \sum_{i\in UHF(S_2)}\kappa_{i1}\kappa_{i2}\frac{1}{2}(\sum_{k=1,2} l_{ik}+\kappa_{ik}d_{ik}) + \kappa_{i1}(1-\kappa_{i2})(l_{i1}+\kappa_{i1}d_{i1}) + (1-\kappa_{i1})\kappa_{i2}(l_{i2}+\kappa_{i2}d_{i2}) + \\
&\quad \sum_{i\in UHF(S_3)}\kappa_{i1}\kappa_{i3}\frac{1}{2}(\sum_{k=1,3} l_{ik}+\kappa_{ik}d_{ik}) + \kappa_{i1}(1-\kappa_{i3})(l_{i1}+\kappa_{i1}d_{i1}) + (1-\kappa_{i1})\kappa_{i3}(l_{i3}+\kappa_{i3}d_{i3}) + \\
&\quad \sum_{i\in UHF(S_4)}\kappa_{i2}\kappa_{i3}\frac{1}{2}(\sum_{k=2,3} l_{ik}+\kappa_{ik}d_{ik}) + \kappa_{i2}(1-\kappa_{i3})(l_{i2}+\kappa_{i2}d_{i2}) + (1-\kappa_{i2})\kappa_{i3}(l_{i3}+\kappa_{i3}d_{i3}) + \\
&\quad \sum_{i\in UHF(S_5)}\kappa_{i1}(l_{i1}+\kappa_{i1}d_{i1}) + \sum_{i\in UHF(S_6)}\kappa_{i2}(l_{i2}+\kappa_{i2}d_{i2}) + \sum_{i\in UHF(S_7)}\kappa_{i3}(l_{i3}+\kappa_{i3}d_{i3}) + \\
&\quad \sum_{i\in UVH(S_2)}\frac{1}{2}\kappa_{i1}\kappa_{i2}(\sum_{k=1,2} l_{ik}+\kappa_{ik}d_{ik}) + \kappa_{i1}(1-\kappa_{i2})(l_{i1}+\kappa_{i1}d_{i1}) + (1-\kappa_{i1})\kappa_{i2}(l_{i2}+\kappa_{i2}d_{i2}) + \\
&\quad \sum_{i\in UVH(S_5)}\kappa_{i1}(l_{i1}+\kappa_{i1}d_{i1}) + \sum_{i\in UVH(S_6)}\kappa_{i2}(l_{i1}+\kappa_{i2}d_{i2}) + \sum_{i\in LVH(S_2)}\kappa_{i1}(l_{i1}+\kappa_{i1}d_{i1})
\end{aligned} \tag{8}
$$

A.4 Network decomposition and Lagrangian relaxation

An interesting observation about the network of stations is that it contains many subnetworks, each corresponding to a physical region, where the interference constraints form cliques. These small subnetworks are linked together but these links are sparse, that is, parts of the network share only a few (or no) nearby stations and hence share few binding constraints from an optimization perspective. For example, consider the reordering of the network so that an interference matrix can be viewed in sub-figure (b) of Figure (11). If we divide the network into two groups, group A with stations 1-381 in the ordered list and group B with the remaining stations, then there is only one pair of stations from A and B that cannot co-share a channel. Without this binding constraint, the problem is separable and can be solved by solving two smaller problems. With the presence of binding constraints, a Lagrangian dual problem can be formulated and solved by using a conjugate gradient method. The method often works well if the number of binding constraints is small (e.g., in the above case with only one constraint)

Figure 11-a shows the adjacency matrix of $G(N,E)$. A dot at row i and column j in Figure 11-a appears if there is potential interference between stations i and j. Figure 11-b show the same interference matrix after we reordered the stations in a way that the non-zeros appear mostly on the diagonals.

We need to divide the original network into $G > 2$ sub-networks. Having larger G implies smaller subproblems and hence there is a better chance that the integer program solver (CPLEX in our experiments) can handle the case. However, having larger G often leads to more binding constraints to be relaxed and hence can beget a larger optimality gap. Thus, we need a good way to divide the network that balances between the sizes of the subproblems and the number of constraints relaxed. Finding a division of the network into two parts with the lowest number of binding constraints is equivalent to the Mincut-Maxflow problem and can be solve efficiently. However, the optimal divisions often contains unbalanced subgraphs with a very large subproblem and the decomposition does not help. Therefore, we need to solve a network partitioning problem to divide the network into G subnetworks with similar sizes.

Suppose we divide the network of stations into G subgroups of stations $\mathcal{S}_1,\ldots,\mathcal{S}_G$. Let us define $\boldsymbol{z}_g$ to be the vector of decision variables that involve only stations in group $\mathcal{S}_g$. The objective function is

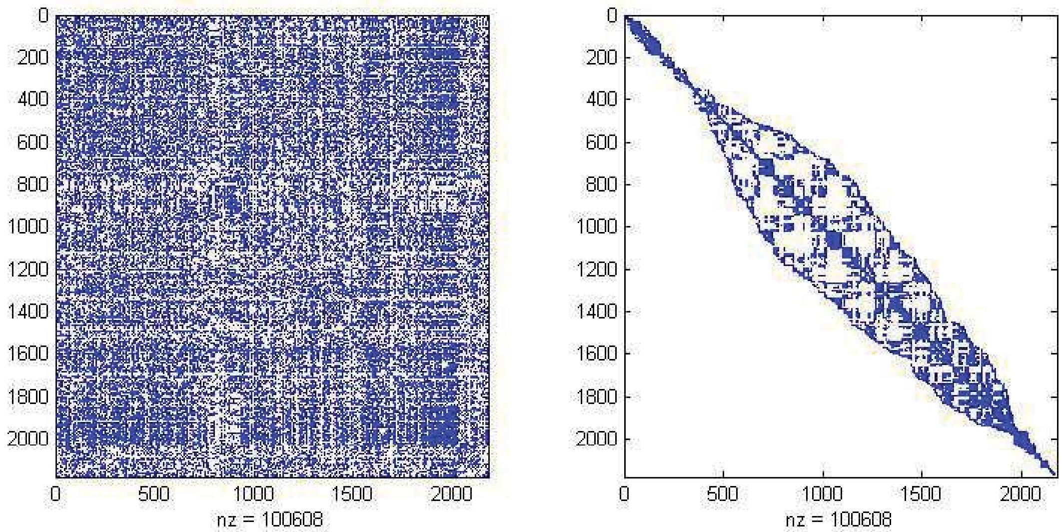


Figure 11: Interference matrix on the original ranking of stations from 1-2177 (left) and after reordering (right).

linear and hence is separable to decision variables $\boldsymbol{z}_g$. The constraint set will contain two groups: (a) constraints that involves only decision variables in one of the subgroups and (b) constraints that involves decision variables from two subgroups. The constraints in (b) corresponds to edges that link stations in different groups. The idea of the Lagrangian relaxation technique is to formulate a Lagrangian dual problem where constraints in (b) are relaxed and are pushed into the objective function. The constraint set is now separable in $\boldsymbol{z}_g$ and the problem is separable for each fixed set of Lagrangian multipliers. In what follows we describe the algorithm in detail.

Let us define

$$\begin{aligned}\mathcal{I}_{cg} &= \{(i,j,k) \in \mathcal{I}_c : i \in \mathcal{S}_g, \}\,,\\ \mathcal{I}_{ag} &= \{(i,j,k) \in \mathcal{I}_a : i \in \mathcal{S}_g, \}\,,\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_{cx} &= \{(i,j,k) \in \mathcal{I}_c : \exists g_1, g_2 \in 1,\ldots,G, g_1 \neq g_2, i \in \mathcal{S}_{g_1}, j \in \mathcal{S}_{g_2}, \}\,,\\ \mathcal{I}_{ax} &= \{(i,j,k) \in \mathcal{I}_a : \exists g_1, g_2 \in 1,\ldots,G, g_1 \neq g_2, i \in \mathcal{S}_{g_1}, j \in \mathcal{S}_{g_2}, \}\,,\end{aligned}$$

Let $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ be the Lagrangian multipliers for co-channel constraints involving set $\mathcal{C}_{cx}$ and for adjacent-channel constraints involving set $\mathcal{C}_{ax}$, respectively. The Lagrangian dual function is

$$\begin{aligned}
\mathcal{L}(\boldsymbol{z}, \boldsymbol{\lambda}, \boldsymbol{\gamma}) &= \sum_{i=1}^{n}(1 - \sum_{k \in \mathcal{C}_i} z_{ik})b_i + \\
&\quad \sum_{(i,j,k)\in\mathcal{I}_{cx}} (1 - z_{ik} - z_{jk})\lambda_{ijk} + \sum_{(i,j,k)\in\mathcal{I}_{cx}} (1 - z_{ik} - z_{jk+1})\gamma_{ijk} \\
&= \sum_{i=1}^{n} |C_i| b_i + \sum_{(i,j,k)\in\mathcal{I}_{cx}} \lambda_{ijk} + \sum_{(i,j,k)\in\mathcal{I}_{ax}} \gamma_{ijk} - \\
&\quad \sum_{i=1}^{n} \sum_{k\in\mathcal{C}_i} z_{ik} \left\{ b_i + \sum_{j:(i,j,k)\in\mathcal{I}_{cx}} \lambda_{ijk} + \sum_{j:(i,j,k)\in\mathcal{I}_{ax}} \gamma_{ijk} + \sum_{j:(j,i,k-1)\in\mathcal{I}_{ax}} \gamma_{ijk} \right\} \\
&= \sum_{g=1}^{G} \mathcal{L}_g(\boldsymbol{z}_g, \boldsymbol{\lambda}, \boldsymbol{\gamma})
\end{aligned}$$

Let us define

$$\mathcal{F}_g(\mathcal{C}) = \left\{ \boldsymbol{z}_g : \begin{array}{l} z_{ik} \in \{0,1\}, \forall i \in \mathcal{S}_g \text{ and } k \in C_i,\ \sum_{k\in C_i} z_{ik} = 1, \forall i \in \mathcal{S}_g, \\ z_{ik} + z_{jk} \le 1, \forall (i,j,k) \in \mathcal{I}_{cg},\ z_{ik} + z_{jk+1} \le 1, \forall (i,j,k) \in \mathcal{I}_{ag} \end{array} \right\}$$

as the set of constraints that involve stations in group $\mathcal{S}_g$, $g = 1, \ldots, G$, only. The constraint set of the relaxed problem is $\{\boldsymbol{z}_g \in \mathcal{F}_g(\mathcal{C}), \forall g = 1, \ldots, G\}$ which is separable in $\boldsymbol{z}_g$. The Lagrangian dual problem is therefore equivalent to

$$\max_{\boldsymbol{\lambda},\boldsymbol{\gamma}} \min_{\boldsymbol{z}\in\mathcal{F}(\mathcal{C})} \mathcal{L}(\boldsymbol{z}, \boldsymbol{\lambda}, \boldsymbol{\gamma}) = \max_{\boldsymbol{\lambda},\boldsymbol{\gamma}} \left[\sum_{g\in 1,\ldots,G} \left\{ \min_{\boldsymbol{z}_g\in\mathcal{F}_g(\mathcal{C})} \mathcal{L}_g(\boldsymbol{z}_g, \boldsymbol{\lambda}, \boldsymbol{\gamma}) \right\} \right]$$

For each fixed set of Lagrangian multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$, the Lagrangian dual problem can be solved by solving G subproblems. If the sizes of the subgroups are reasonable such that the subproblems can be solved efficiently, we can apply a sub-gradient method to find the optimal set of the Lagrangian multipliers. The final solution provides us an upper bound to the original problem.

Optimizing Prices in Descending Clock Auctions*

Tri-Dung Nguyen† Tuomas Sandholm ‡§

Abstract

A descending (multi-item) clock auction (DCA) is a mechanism for buying items from multiple potential sellers. In the DCA, bidder-specific prices are decremented over the course of the auction. In each round, each bidder might accept or decline his offer price. Accepting means the bidder is willing to sell at that price. Rejecting means the bidder will not sell at that price or a lower price. DCAs have been proposed as the method for procuring spectrum from existing holders in the FCC's imminent incentive auctions so spectrum can be repurposed to higher-value uses. As prior papers on DCAs, in this paper we focus on the case where a bidder has one option: to stay or get off the air.

However, the DCA design has lacked a way to determine the prices to offer the bidders in each round. This is a recognized, important, and timely problem.

We present, to our knowledge, the first (in the March 2014 FCC docket version and the June 2014 EC conference version of this paper) techniques for this. We develop a percentile-based approach which provides a means to naturally reduce the offer prices to the bidders through the bidding rounds. We also develop a continuous optimization model for setting prices so as to minimize expected payment while stochastically satisfying the feasibility constraint. (The DCA has a final adjustment round that obtains feasibility after feasibility has been lost in the final round of the main DCA.) We prove attractive properties of this, such as symmetry and monotonicity. We develop computational methods for solving the model, and develop certain kinds of clique constraint for the continuous optimization model that improve the results further by more accurate handling of feasibility. (We also develop optimization models with recourse, but they are not computationally practical.)

We present experiments both on the homogeneous items case and the case of FCC incentive auctions, where we use real interference constraint data to get a fully faithful model of feasibility. The optimization-based price-setting approach significantly outperforms the natural percentile-based approaches in minimizing the final payment by the auctioneer. It also helps feasibly repacking more stations. An unexpected paradox about DCAs is that sometimes when the number of rounds allowed increases, the final payment increases. We provide an explanation for this.

Keywords: Combinatorial Auction; Descending Clock Auction; Incentive Auction; Spectrum Auction; Winner Determination; Pricing; Optimization.

1 Introduction

A *descending (multi-item) clock auction* (DCA) is a mechanism for buying items from multiple potential sellers. In the DCA, bidder-specific prices are initialized at reserve prices and then decremented over the course of the auction [18]. In each round, the auctioneer decrements the offer prices to the bidders, who might accept or decline the offers. Accepting means that the bidder is willing to sell at that price. Rejecting means that the bidder has to exit the auction permanently and cannot sell. This process is repeated until

*Patent pending. A short early version of this paper appeared in the Proceedings of the *ACM Conference on Economics and Computation (EC)*, Palo Alto, CA, June 2014.

†Professor, University of Southampton, School of Mathematics.

‡Professor, Carnegie Mellon University, Computer Science Department; email: sandholm@cs.cmu.edu.

§Corresponding author.

the auctioneer's target number of items to purchase would become infeasible if additional bidders were to reject any new (lower) offers. At that point the auction ends and the current prices are paid.

We consider the following setting. The auctioneer wants to buy items from a pool, N, of n potential sellers. Each seller $i \in N$ has a specific type of item G_i and decides to sell it or not depending on the offer price. The items from the sellers could be substitutable and complementary. The auctioneer has a target number of items to buy, T, and there is a feasibility function $F : 2^N \to \{0, 1\}$ that specifies, for each subset of potential sellers, S, whether the items from S can fulfill the target T or not; i.e. $F(S) = 1$ if the combined items from $\{G_i, i \in S\}$ fulfill the target.

A simple example of this is the case where the sellers have identical items and the auctioneer wants to buy a target number T them. In this case, the feasibility is simply $F(S) = 1$ if $|S| \geq T$, and $F(S) = 0$ otherwise. However, in real-world application such as the FCC spectrum reverse auctions discussed later, the feasibility function can be highly complex. Often it cannot be given in closed form, but rather is stated through constraints as an optimization problem.

In the DCA, the auctioneer sends offer prices to the sellers and checks whether they accept those prices. Bidders who accept the offers are called *active*. If the combined items from the active bidders fulfill the target, then the auctioneer reduces the prices further in the next round and repeats the process. If at some point the items from the active bidders do not fulfill the target, then the auctioneer goes back to the last step and conducts a last-round adjustment to offer higher prices to some declined bidders so that feasibility is obtained. Algorithm 1 describes the general DCA framework.

ALGORITHM 1: Descending Clock Auction (DCA)

Input: A set of sellers $\mathcal{N} = \{1, \ldots, n\}$ with goods $\{G_1, \ldots, G_n\}$, an auctioneer with a feasibility mapping function $F : 2^{\mathcal{N}} \to \{0, 1\}$. A target number of rounds allowed m. Initial value function estimates v_i.

Output: A set of feasible sellers $\mathcal{A} \subset \mathcal{N}$, i.e., $F(\mathcal{A}) = 1$, and the corresponding vector of offer prices $\boldsymbol{p}$ that aims to minimize the expected payment that the buyer needs to pay.

1. Initialize the price vector $\boldsymbol{p}$ to the reserve prices. Let the set of active bidders be $\mathcal{A}^{(0)} = \mathcal{N}$;

for *round* $r = 1...m$ **do**

  2.1. Find a vector of prices $\boldsymbol{p}$ to offer the bidders;

  2.2. Find the set of rejected offers $\mathcal{R}$;

  if $F(\mathcal{A}^{(r)} \backslash \mathcal{R}) = 1$ **then**

    2.2.1. $\mathcal{A}^{(r+1)} \leftarrow \mathcal{A}^{(r)} \backslash \mathcal{R}$;

    2.2.2. Update the distributions of the bidders' values;

  else

    2.2.3. Enter the readjustment round in Step 3;

  end

end

3. Readjust the prices for bidders in the last round to meet the target;

4. Pay winning bidders the offered prices;

A key challenge in a DCA lies in how to set the prices offered to the bidders. A natural approach is to set the offer price at some fixed percentile of the (buyer's model of the) distribution of that bidder's valuation. For example, the prices could be set so that each bidder has the same probability of accepting her offer. The choice of the percentile would depend on what the auctioneer aims for on the trajectory of the sizes of sets of active bidders through the rounds. An example of the trajectory could be such that the expected number of rejections in each round is distributed evenly throughout the auction. Another example of the trajectory is to set a fixed percentage of rejection in each round, that is, the expected number of rejections is proportional to the size of the remaining set of active bidders. We call this class of methods *percentile based* and we will describe it formally later in Section 4.1.

Those methods have several drawbacks. First, having a fixed percentile means there is no way to distinguish between bidders with high bid values and the rest; hence the final payment will likely be unnecessarily high due to the probabilistic inclusion of high-priced bidders. More importantly, those methods do not have any special treatment for the degree of interaction among the items in the feasibility function.

This paper presents a significant improvement within the DCA by addressing these issues by designing an optimization model for setting the prices. The model is designed to minimize the expected final payment while ensuring feasibility in a stochastic sense. It is flexible in that it can incorporate bidder-specific characteristics with respect to feasibility.

1.1 FCC Incentive Auctions

The current flagship application of DCAs is the Federal Communications Commission (FCC) *incentive auctions*. The FCC has been selling radio spectrum licenses via auctions since 1994 [4, 17]—in recent years via combinatorial auctions [6, 7]. However, there is not enough spectrum left to sell for the new high-value spectrum uses that have arisen. The idea of incentive auctions, therefore, is to buy some of the existing licenses back from their current holders, which frees up spectrum, and then to sell spectrum to higher-value users. The idea of such incentive auctions was introduced in the 2010 National Broadband Plan [9]. It is motivated by the fact that the demand and the value of over-the-air broadcast television has been declining while the demand for mobile broadband and wireless services has increased dramatically in recent years. Given the limited spectrum resources, incentive auctions were introduced as a voluntary, market-based means of repurposing spectrum. This is done by creating a market that exchanges the usage rights among the two groups of users: (a) existing TV broadcasters and (b) wireless broadband networks. Three key players in this market are existing spectrum owners, spectrum buyers, and the FCC, which acts as the intermediator.

An incentive auction consists of three stages [12] (see also a whitepaper about design choices by Hazlett et al. [14]):

1. *Reverse auction*: parts of the spectrum currently used by TV broadcasters is bought back.
2. *Repacking*: remaining broadcasters are reallocated to a smaller spectrum band.
3. *Forward auction*: freed spectrums is sold via a (combinatorial) auction for use in wireless broadband networks.

In the reverse auction, we need to find a set of stations to be reallocated to lower-band channels and a separate set of stations to be bought off the air, in order to achieve the following goals: (a) meet some target on the number of contiguous channels freed on the higher spectrum band and (b) minimize total payment by the FCC. The FCC is required to respect the broadcasters' carry-right, which means, in the context of a DCA, a station that rejects the offer still has the right to stay on the air, but possibly on a lower spectrum band. This repacking stage needs to ensure that all the stations that rejected their offers can be feasibly repacked into the allocated band without violating the engineering constraints, that is, interference-free population coverage, as we will detail later in the paper.

Once the reverse auction phase is completed and the remaining stations are repacked, the FCC announces the cleared spectrum which is now available for purchase. Buyers then submit bids on bundles of spectrum frequencies. The FCC solves a winner determination problem to decide which bids to accept or reject. This does not involve the engineering constraints, so it is like a standard combinatorial auctions; hence existing winner determination algorithms (e.g., [22, 23, 24, 25, 8, 21]) can be used for this. Therefore, we focus on the reverse auction stage.

The FCC aims to free up a number of high spectrum channels—say between channel 32 and 51. In order to do that, all stations need to be assigned to the lower band channels or give up their carry right in exchange for some payment. The process of deciding which stations to retain, which stations to leave, and at what prices, is done through the reverse auction. Within this stage, the repacking problem needs to be solved in order to check whether the remaining stations can be feasibly reassigned to the targeted lower-band channels. Other groups have recently also tackled the repacking part (e.g., Leyton-Brown [16]).

Two apparent options for the reverse auction design are a single-round Vickrey-Clarke-Groves (VCG) auction or a DCA.[1] In the VCG, stations submit sealed bids. The auctioneer then solves the winner determination problem (WDP) to determine the winning bids, that is, those stations that the FCC buys back at their bid prices. Stations with rejected bids are repacked into the lower-band channels. Unfortunately, solving this WDP is challenging because it involves thousands of binary variables and millions of interference-avoidance constraints. In fact, the FCC has attempted to solve it, but according to Milgrom and Segal [19], solving an instance of this problem with state-of-the-art optimization packages takes weeks of compute time without finding the optimal solution. Also, even if the list of winners were available, solving for the VCG payment for each winning bidder would involve solving a combinatorial problem similar to the WDP. It is the same as the WDP except that a small portion of the binary variables are fixed. Furthermore, a small approximation error in solving these problems can lead to significant over-payment [18].

The DCA was proposed by Milgrom and Segal [18], who have shown attractive features of the framework. According to documents that the FCC uses to communicate with its stakeholders regarding incentive auctions (wireless.fcc.gov/ incentiveauctions/learn-program), the DCA is the FCC's design of choice.

However, there was a key missing piece in the DCA: how to determine what prices to offer to the different bidders in the different rounds. In this paper we present a solution to this. To our knowledge, this is the first paper on the topic.[2]

1.2 Contributions

The main contributions of this work include the following.

- We develop a natural percentile-based approach and optimization models for setting offer prices in DCAs. We study analytical properties of the optimal solutions, as well as the computational methods and their complexity. To the best of our knowledge, this is the first paper to investigate this topic.
- We provide numerical results for simulated auctions as well as for the actual ongoing incentive auctions using real data. This is a very large-scale auction—both in the dollar value and the complexity of the auctions.
- We show that our optimization models support flexibility in incorporating problem-specific information. In the case of incentive auctions, adding certain kinds of clique constraint to the continuous optimization problem captures feasibility better and yields better results (and reduces the possibility of collusion among bidders). The advanced optimization-based approach significantly outperforms the natural percentile-based approach both in terms of final payments and the number of stations being feasibly repacked.

[1] Combinatorial reverse auctions have been extensively used for sourcing of various categories of goods and services in industry and government [24, 20]. Typically, a pricing rule has been used where bidders pay the prices offered by them in their winning bids. Normally, one to three rounds of bidding have been used, with feedback to bidders in between rounds, but also continuous variants have been used for small numbers of items in the auction, where tentative winner determination is conducted each time a bid is submitted or revised [24]. Feedback and bidder strategies in combinatorial auctions have also been studied in laboratory experiments [1]. Combinatorial reverse auctions have also been proposed for sourcing carrier-of-last-resort responsibility for universal service [15]. The descending clock auction is a different design where the prices are non-combinatorial, that is, only individual items are priced.

[2] We submitted it to the FCC docket on March 25th, 2014 and presented it at the EC conference in June 2014.

2 Techniques for Optimizing Offer Prices in the Descending Clock Auction

A key component of a DCA is to set the prices to offer to the active bidders. The auctioneer needs to consider the tradeoff between minimizing payment to the accepted bidders and fulfillment of the target (i.e., repacking feasibility in the case of incentive auctions).

Furthermore, the pricing affects the speed of the auction in terms of the number of rounds. Therefore, there is another tradeoff. On the one hand, if the offer prices are too high, many rounds are required, and that may be undesirable from the perspective of minimizing logistical effort. On the other hand, if the offer prices are too low, many bidders reject and the auction ends too quickly without properly serving its price-discovery purpose.

In this section, we present a method for setting the offer prices. They should be dependent on (a) the estimated value functions of the bidders, (b) the importance of the items for the target to be fulfilled, and (c) how quickly/slowly the auctioneer wants the auction to run. We provide an optimization model that incorporates these considerations.

Throughout this section, we assume the auctioneer's target T is a given scalar number and the feasibility mapping F is given in some specific form. Obviously these inputs are not readily available in many settings such as in incentive auctions: the feasibility function involves a complex repacking problem. Later in Section 3 we show how to translate such a complex setting to the forms of T and F described in this section.

2.1 Optimization Model for Price Setting in Each Round

Suppose each bidder has a threshold price v_i below which it declines the offer. The auctioneer does not know these threshold prices. Suppose, however, that the auctioneer has an estimate/belief that the threshold price v_i of bidder i follows some distribution on the support set $[l_i, u_i]$. Assume that the auctioneer knows these distributions.

Let $X_i(p_i)$ be the corresponding Bernoulli random variable that indicates whether bidder i will accept the offer at price p_i. The total payment is $c(\boldsymbol{p}) = \sum_{i \in \mathcal{A}^{(r)}} X_i(p_i) p_i$, where $\mathcal{A}^{(r)}$ is the set of remaining active bidders in the current round r. One objective of the auctioneer is to set the prices in such a way that minimizes the expected payment which is equal to

$$E[c(\boldsymbol{p})] \quad = \sum_{i \in \mathcal{A}^{(r)}} E[X_i(p_i)] p_i. \tag{1}$$

The expected payment is nondecreasing in the offer prices. So, setting low prices would lead to low expected payment. However, doing this would lead to a high chance of reaching infeasibility too soon, so little price discovery could be done. The auctioneer, therefore, needs to balance feasibility with low expected payment. The bidders' values are random to the auctioneer, so, for any fixed set of offer prices, the feasibility can only be expressed in a stochastic sense. It is possible to apply ideas from stochastic programming to model this as a chance constraint. However, this often leads to computational intractability since the feasibility mapping tends to be a highly complex function of the prices. We propose a simpler measure of the expected number of bidders accepting the offers $N(\boldsymbol{p})$ as this is directly related to the chance of feasibility, i.e., the larger the population of active bidders, the higher the chance of feasibility. We have $E[N(\boldsymbol{p})] = \sum_{i \in \mathcal{A}^{(r)}} E[X_i(p_i)]$. The problem of minimizing the expected payment while ensuring the expected number of accepted bidders to meet some target can be modeled as the following optimization problem:

$$\begin{aligned} \min_{\boldsymbol{p}} \quad & \sum_{i \in \mathcal{A}^{(r)}} E[X_i(p_i)]p_i, \\ \text{s.t.} \quad & \sum_{i \in \mathcal{A}^{(r)}} E[X_i(p_i)] \geq T^{(r)}, \\ & l_i \leq p_i \leq u_i, \forall i \in \mathcal{A}^{(r)}, \end{aligned} \tag{2}$$

where $T^{(r)}$ is the targeted number of active bidders at round r. The auctioneer has the flexibility in choosing this target depending on how quickly or slowly the auctioneer wants to complete the auction. One possibility is to set

$$T^{(r)} = n^{(r)} - \frac{n^{(r)} - T}{m - r + 1},$$

where $n^{(r)}$ is the number of active bidders remaining, T is the final target, and $(m - r + 1)$ is the number of remaining rounds. In this case, $T^{(r)}$ is set such that the size of the sets of active bidders reduces evenly throughout the rounds.

Both the payment and feasibility are expressed in expectation due to the stochasticity of the bidders' values. This means the actual number of accepted bidders might exceed or fall below the target $T^{(r)}$. That potential disparity, and its effect on the later rounds, imply that better prices than those from Model 2 could be obtained via a model that incorporates recourse. We present one such model in Appendix A.2. That model is, however, computationally highly complex to solve and might not be appropriate in large-scale DCA settings if quick rounds are needed.

The price-setting strategy in Model 2 simplifies one important fact about the dynamical nature of DCAs: the choice of offer prices in round r will affect the population of active bidders in subsequent rounds as well as the distribution of those bidders' values. Thus, in principle, the pricing problem in DCAs should be modeled as a dynamic program as presented in Appendix A.1. Solving it, however, would be prohibitively complex. Instead, we simplify this process through a dynamic scheduling of the sizes of the sets of active bidders $T^{(r)}$. Specifically, we schedule the size of the set of active bidders $n^{(r)}$ evenly in the last $(m-r+1)$ rounds. The simplicity of the models described in this section makes them appropriate in situations when it is critical to offer prices to the bidders in a timely manner.

To simplify the notation in Model 2 a bit, let us denote by $F_i(p_i)$ the cumulative distribution of the valuation of bidder i. It can also be interpreted as the probability that bidder i will accept offer price p_i, i.e., $F_i(p_i) \equiv \delta_i \equiv E[X_i(p_i)]$. Model 2 can now be rewritten as

$$\begin{aligned} \min_{\boldsymbol{p}} \quad & \sum_{i \in \mathcal{A}^{(r)}} F_i(p_i)p_i, \\ \text{s.t.} \quad & \sum_{i \in \mathcal{A}^{(r)}} F_i(p_i) \geq T^{(r)}, \\ & l_i \leq p_i \leq u_i, \forall i \in \mathcal{A}^{(r)}. \end{aligned} \tag{3}$$

2.1.1 Uniform Distribution on Bidder Values

We first consider the case where the bidders' values are drawn from uniform distributions on $[l_i, u_i]$. The probability that bidder i will accept the offer price v_i is,

$$\delta_i(p_i) = F_i(p_i) = \begin{cases} 0, & \text{if } p_i \leq l_i, \\ \frac{p_i - l_i}{u_i - l_i}, & \text{if } l_i \leq p_i \leq u_i, \\ 1, & \text{if } p_i > u_i. \end{cases} \tag{4}$$

Naturally, we can restrict the price to $l_i \leq p_i \leq u_i$. The expected payment is

$$E[c(\boldsymbol{p})] = \sum_{i \in \mathcal{A}^{(r)}} F_i(p_i) p_i = \sum_{i \in \mathcal{A}^{(r)}} \frac{(p_i - l_i) p_i}{u_i - l_i}.$$

The constraint on the expected number of stations accepting the offers is

$$E[N(\boldsymbol{p})] \geq T^{(r)} \quad \Leftrightarrow \quad \sum_{i \in \mathcal{A}^{(r)}} \frac{p_i - l_i}{u_i - l_i} \geq T^{(r)},$$

which is linear in the offer prices $\boldsymbol{p}$. So, the problem of minimizing the expected payment while ensuring the probabilistic constraints on feasibility can be modeled as the following quadratic program:

$$\begin{aligned} \min_{\boldsymbol{p}} \quad & \sum_{i \in \mathcal{A}^{(r)}} \frac{(p_i - l_i) p_i}{u_i - l_i}, \\ \text{s.t.} \quad & \sum_{i \in \mathcal{A}^{(r)}} \frac{p_i - l_i}{u_i - l_i} \geq T^{(r)}, \\ & l_i \leq p_i \leq u_i, \forall i \in \mathcal{A}^{(r)}. \end{aligned} \tag{5}$$

The problem has a strictly convex separable quadratic objective and one joint constraint in addition to lower and upper bound constraints. Thus, the problem has a unique optimal solution. In addition, the expected payment increases with $\boldsymbol{p}$ while the expected number of rejected stations decreases with $\boldsymbol{p}$. Thus, we would expect the constraint to be tight at the optimal solution. Let λ be the Lagrangian multipliers for the joint constraint $\sum_{i=1}^{n} \frac{p_i - l_i}{u_i - l_i} \geq T^{(r)}$. The Lagrangian dual function is

$$\begin{aligned} \mathcal{L}(\lambda, \boldsymbol{p}) &= \sum_{i \in \mathcal{A}^{(r)}} \frac{(p_i - l_i) p_i}{u_i - l_i} + \lambda \left(T^{(r)} - \sum_{i \in \mathcal{A}^{(r)}} \frac{p_i - l_i}{u_i - l_i} \right) \\ &= T^{(r)} \lambda + \sum_{i \in \mathcal{A}^{(r)}} \frac{(p_i - l_i) p_i - \lambda (p_i - l_i)}{u_i - l_i}. \end{aligned}$$

The Lagrangian dual problem can be derived as

$$\max_{\lambda \geq 0} \quad \left\{ T\lambda + \sum_{i \in \mathcal{A}^{(r)}} \min_{l_i \leq p_i \leq u_i} \frac{p_i^2 - (\lambda + l_i) p_i + \lambda l_i}{u_i - l_i} \right\}.$$

The problem is convex, so there is no duality gap when we take the Lagrangian relaxation. For each fixed set of Lagrangian multipliers $\boldsymbol{\lambda}$, the optimal prices can be derived:

$$p_i^* = \begin{cases} \frac{l_i + \lambda}{2}, & \text{if } l_i \leq \lambda \leq 2u_i - l_i, \\ l_i, & \text{if } \lambda \leq l_i, \\ u_i, & \text{otherwise.} \end{cases} \tag{6}$$

Proposition 1. *[Symmetry] For any two bidders with the same valuation distribution, the optimal offer prices must be the same.*

Proof. Straightforward from Formulation 6. For any two bidders i, j with the same boundaries, i.e., $l_i = l_j$ and $u_i = u_j$, any choice of λ leads to $p_i^* = p_j^*$. $\square$

The Lagrangian dual function can be calculated efficiently in $O(n)$ for each fixed λ. The Lagrangian dual problem is a piece-wise concave maximization problem with one scalar variable on the non-negative orthant. One could apply a conjugate gradient method to solve this problem. Better still, we present an algorithm that exploits the structure of the problem and is $\mathrm{O}(n \log n)$:

Proposition 2. *The optimal offer prices can be found—i.e., Model 5 can be solved—in $\mathrm{O}(n \log n)$ operations.*

Proof. The optimal offer price p_i^* is a piece-wise linear function on λ with three pieces that are intersected at two points l_i and $(2u_i - l_i)$. We can order all the $2n$ points $\{l_i, 2u_i - l_i\}, \forall i = 1, \ldots, n$ on the vertical axis in $\mathrm{O}(n \log n)$. From that we obtain $(2n+1)$ pieces (some potentially with zero length). For λ that falls within each piece, we have a corresponding linear mapping to the offer prices $p_i, i = 1, \ldots, n$. Thus, the Lagrangian function on that piece can be calculated as a corresponding quadratic function on λ. The optimum λ on each piece can therefore be calculated. The global optimal λ can be found by taking the maximum Lagrangian function among all the $(2n+1)$ pieces. $\square$

2.1.2 Heterogeneous Bidders with Respect to Target Feasibility

The feasibility constraint in Model 5 was for the homogeneous case. Consider now a heterogeneous case where the item from each bidder would affect feasibility differently. For example, in incentive auctions, depending on the current list of rejected stations, the repacking feasibility is highly sensitive to the choice of new rejection due to engineering constraints. Suppose there is a weight vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ that represents the contribution from the bidders to the feasibility function F; in Section 3 we will show how to derive these weights for the case of incentive auctions. For now, we consider a simple example where bidders all share the same type of item but each of them will have a different number of items to sell. In that case, ω_i can represent the amount available from bidder i. We assume a setting where each bidder considers its item as a single product and considers selling it or not depending solely on the offer price (to avoid the combinatorial complexity). Now, Model 5 can be modified to

$$\begin{aligned} \min_{\boldsymbol{p}} \quad & \sum_{i \in \mathcal{A}^{(r)}} \frac{(p_i - l_i) p_i}{u_i - l_i}, \\ \text{s.t.} \quad & \sum_{i \in \mathcal{A}^{(r)}} \omega_i \frac{p_i - l_i}{u_i - l_i} \geq T^{(r)}, \\ & l_i \leq p_i \leq u_i, \forall i \in \mathcal{A}^{(r)}. \end{aligned} \tag{7}$$

Applying the same Lagrangian relaxation method as shown in solving Model 5,

$$p_i^* = \begin{cases} \frac{l_i + \omega_i \lambda}{2}, & \text{if } l_i/\omega_i \leq \lambda \leq (2u_i - l_i)/\omega_i, \\ l_i, & \text{if } \lambda \leq l_i/\omega_i, \\ u_i, & \text{otherwise.} \end{cases} \tag{8}$$

Similar to Model 6, the problem is strictly convex, so it has a unique optimal solution and there is no duality gap when we take the Lagrangian relaxation.

Proposition 3.

a) *[Symmetry] For any two bidders with the same valuation distribution and with the same weights, the optimal offer prices must be the same.*

b) *[Monotonicity] For any bidders with the same valuation distribution, the optimal offer prices are higher for those with higher weights.*

Proof. In part (a), for any two bidders with identical value distributions and weights, any choice of λ would lead to the same formulations for the offer prices (as shown in Formulation 8) for the two bidders, which means the optimal offer prices are the same. Part (b) can be derived in the same way with a note that both ω and λ are non-negative. □

The implication of part (b) is that the auctioneer should offer higher prices to more 'important' bidders to keep them active. We will show how this is applied to the case of incentive auction where the stations affect the feasibility differently.

Proposition 4. *The optimal offer prices can be found—i.e., Model 7 can be solved—in* $\mathrm{O}(n \log n)$ *operations.*

Proof. The proof is similar to that of Proposition 2. □

2.1.3 General Valuation Distributions

Now we drop the assumption that the bidders' valuation distributions are uniform. In this setting, Model 7 generalizes to a nonlinear program:

$$\begin{aligned} \min_{\boldsymbol{p}} \quad & \sum_{i \in \mathcal{A}^{(r)}} F_i(p_i) p_i, \\ \text{s.t.} \quad & \sum_{i \in \mathcal{A}^{(r)}} \omega_i F_i(p_i) \geq T^{(r)}, \\ & l_i \leq p_i \leq u_i, \forall i \in \mathcal{A}^{(r)}. \end{aligned} \tag{9}$$

Depending on the valuation distributions, this program might be non-convex and might be hard to solve in general. The attractive part of this model, however, is that it has a separable objective function and only one joint constraint. Using the same Lagrangian relaxation method, we obtain the following Lagrangian dual problem:

$$\max_{\lambda \geq 0} \quad \left\{ T^{(r)} \lambda + \sum_{i \in \mathcal{A}^{(r)}} \min_{l_i \leq p_i \leq u_i} \left(F_i(p_i) p_i - \lambda \omega_i F_i(p_i) \right) \right\}.$$

For each fixed Lagrangian multiplier λ, the optimal prices $\boldsymbol{p}$ of the inner problem can be found by solving n nonlinear sub-problems, each with a single scalar variable. In the case where the distribution of the value function is piecewise linear, the inner problem is a piecewise cubic function where the optimal solution of each piece can be found efficiently.

The problem might be non-convex so the Lagrangian relaxation method might produce some optimality gap. The Lagrangian relaxation method has been shown to perform well empirically, that is, with small optimality gaps, for a number of combinatorial problems (see Fisher [13] for details about the method and successful applications). The offer prices found, however, might not be optimal. In what follows we present a method that could either be used as a stand-alone method for finding the optimal offer prices or can be used in combination with the Lagrangian relaxation method to enhance the performance. This method takes advantage of the discreteness property of the decision variables, the separability of the objective function, and the single joint constraint. Consider an auction design where the offer price for bidder i must take discrete values in a given set $\{P_{i1}, \ldots, P_{ik}\}$. This restriction often holds in practical auction settings because (1) allowing fractional bids can increase hassle (e.g., bookkeeping), and (2) allowing unimportant digits to be expressed opens the door for collusion among bidders. Such collusion has been observed in FCC auctions and the FCC has subsequently practiced, for example, "click-box" bidding where bidders have to select their bids from a small set of discrete values [5, 2]. The problem becomes

$$\begin{aligned}
\min_{\boldsymbol{p}} \quad & \sum_{i=1}^{n} F_i(p_i)p_i, \\
\text{s.t.} \quad & \sum_{i=1}^{n} \omega_i F_i(p_i) \geq T, \\
& p_i \in \{P_{i1}, \ldots, P_{ik}\}, \forall i = 1, \ldots, n.
\end{aligned} \tag{10}$$

This is a mixed-integer nonlinear program. They are generally very difficult to solve. However, this one has a knapsack-type structure so a dynamic program can be utilized:

Proposition 5. *Optimal offer prices for Model 10 can be found in* $\mathrm{O}(KLn^2)$ *operations, where* K *is the maximum number of discrete price levels and* L *is the number of precision points in the range* $[0, 1]$ *used to calculate the cumulative values* $F_i(\cdot)$*.*

The proof of Proposition 5 is presented in Appendix A.3. Here we discussed the problem in the first round where $\mathcal{A}^{(1)} = \mathcal{N}$ and has the largest size. Results for other rounds can be derived similarly. Also, it is possible to combine the idea from Lagrangian relaxation and dynamic programming to improve the computational performance further. This can be done by using the approximated results from the Lagrangian relaxation method and using it to guide the discretization of the feasible space for $\boldsymbol{p}$ before applying the dynamic program. For example, the discretization around the solution suggested by the Lagrangian relaxation could be more refined than elsewhere.

2.2 Updating Value Function Distributions

The auctioneer utilizes the price discovery feature of the DCA to update the estimated value functions in Step 2.2.2 of Algorithm 1. Suppose that, at the beginning of round r, the auctioneer has a belief that the random threshold price v_i for bidder i follows a distribution on the support set $[l_i, u_i]$ with a cumulative distribution $F_i(\cdot)$. Once the auctioneer has made an offer $p_i^{(r)}$ to i, there are two cases. In the first case, i rejects the offer. The rejected list is updated and i is no longer active (except in the last round when the target is not met). In the second case, i accepts the offer and stays active. The auctioneer needs to update the belief about the threshold price of bidder i based on the fact that the bidder accepted. We need to find the conditional distribution for $(v_i \mid v_i \leq p_i^{(r)})$, which can be calculated as follows:

$$Prob(v_i \leq a \mid v_i \leq p_i^{(r)}) = \begin{cases} 0, & \text{if } a \leq l_i, \\ 1, & \text{if } a > p_i^{(r)}, \\ \frac{Prob(v_i \leq a)}{Prob(v_i \leq p_i^{(r)})} \equiv \frac{F_i(a)}{F_i(p_i^{(r)})}, & \text{if } l_i \leq a \leq p_i^{(r)}. \end{cases} \tag{11}$$

The auctioneer then uses this new set of beliefs on the value functions and the new set of active bidders to run a new auction round.

If $F_i(p_i)$ is piece-wise linear at iteration r, then the updated cumulative distribution is also piece-wise linear at iteration $(r+1)$. In the special case of a uniform distribution, the updated distribution is uniform on $[l_i, p_i^{(r)}]$, that is, we simply update the upper bound to be $p_i^{(r)}$.[3]

[3] In general, there could also be other factors to take into account when updating the value function distribution. For one, the auctioneer might want to update the bounds based on how other bidders have responded to offers. This, and DCAs in general, beget interesting questions for future research related to interdependent valuations. Within the scope of this paper, our focus is, however, on how to set offer prices given the beliefs.

2.3 Final-Round Settlement

In Step 3 of Algorithm 1, the auctioneer selects the winners of the auctions after having undertaken the price discovery through the multiple rounds of the DCA. Let $\boldsymbol{p}$ be a vector of prices that the auctioneer offers to the bidders in the last round. Given the offer prices, the bidders might accept or reject the offers. The auctioneer then updates the upper bounds on the bidders' values: upper bounds for bidders that accepted will be updated to the offer prices and those of the bidders who rejected remain unchanged. After the final round, the auctioneer does not have any further opportunity to conduct price discovery, and has to decide which bidders are the final winners to meet the target. In the simple setting where the auctioneer's target is to obtain T items out of a pool of n homogeneous items, the auctioneer chooses T bids with the smallest updated upper bounds and pays each of those bidders that price. For the weighted case, the auctioneer solves the following knapsack problem to determine a set of winners that meets the target with the least total payment (recall that u_i is the most up-to-date upper bound on the valuation of i):

$$\begin{aligned} \min_{\boldsymbol{z}} \quad & \sum_{i \in \mathcal{A}^{(m)}} u_i z_i, \\ \text{s.t.} \quad & \sum_{i \in \mathcal{A}^{(m)}} w_i z_i \geq T, \\ & z_i \in \{0,1\}, \forall i \in \mathcal{A}^{(m)}, \end{aligned} \tag{12}$$

where z_i indicates whether bid i should win.

2.4 DCA with Homogeneous Items

In this section we study a relatively simple auction with n bidders, each of whom has one item to be sold. The objects are identical from the auctioneer's point of view, and the auctioneer has a target of buying T objects. In this case, the feasibility function is $F(\mathcal{A}) = 1$ if $|\mathcal{A}| \geq T$, and $F(\mathcal{A}) = 0$ otherwise.

To apply the general DCA framework of Algorithm 1 to this relatively simple setting, we need to adapt Algorithm 1 in Step 2.1 and set the target number of accepting bidders to $T^{(r)} = n^{(r)} - (n^{(r)} - T)/(m - r + 1)$, where $n^{(r)} = |\mathcal{A}^{(r)}|$, and solve Model 9 to find a vector of prices $\boldsymbol{p}$ to offer the bidders. In this step, we assume the auctioneer first aims for a trajectory of the sets of active bidders and then optimizes the prices correspondingly. Another strategy that the auctioneer could adopt is to first try to 'foresee' the offer prices in the final round by solving Model 9 for $T^{(r)} = T$, and then shrink these final prices to the upper bound by taking advantage of the multi-round guessing in price discovery. We present details about this strategy in Appendix A.4. We also need to adapt Step 3 of Algorithm 1 by solving Formulation 12. Details about this and other steps are presented in Algorithm 2.

3 DCA in FCC Incentive Auctions

In this section we conduct experiments with a model of the FCC incentive auction that uses real FCC data regarding the feasibility. A key feature of reverse auctions in incentive auctions is that the feasibility function is highly sensitive to the set of rejected bids. This is due to a large set of engineering constraints between the stations that restricts them from being assigned to the same or adjacent channels. This means the inclusion of a set of stations in a current reject list would make the characteristics of remaining stations totally different from each other. We first describe these interference constraints and the feasibility function. We then show how the general DCA framework can be applied to incentive auctions.

ALGORITHM 2: A DCA Framework using Optimal Price Setting for the Homogeneous Setting

Input: A set of sellers $\mathcal{N} = \{1, \ldots, n\}$ with goods $\{G_1, \ldots, G_n\}$, an auctioneer with a target T. A target number of rounds allowed m. Initial valuation estimates v_i.

Output: A set of feasible sellers $\mathcal{A} \subset \mathcal{N}$, i.e., $|\mathcal{A}| = T$, and the corresponding offer price vector $\boldsymbol{p}$ that aims to minimize the expected payment.

1. Let the set of active bidders be $\mathcal{A}^{(r)} = \mathcal{N}$;

for *round* $r = 1...m$ **do**

 2.1. Set the target number of accepting bidders $T^{(r)} = n^{(r)} - (n^{(r)} - T)/(m - r)$ where $n^{(r)} = |\mathcal{A}^{(r)}|$ and solve Model 9 to find a vector of prices $\boldsymbol{p}$ to offer the bidders;

 2.2. Find the set of rejected offers $\mathcal{R}$;

 if $|(\mathcal{A}^{(r)} \backslash \mathcal{R}| \geq T$ **then**

  2.2.1. $\mathcal{A}^{(r+1)} \leftarrow \mathcal{A}^{(r)} \backslash \mathcal{R}$;

  2.2.2. Update the distributions of the bidders' valuations using Formulation 11;

 else

  2.2.3. Enter the adjustment round in Step 3;

 end

end

3. Adjust the prices for bidders in the last round to meet the target by solving Formulation 12;

4. Pay winning bidders the offer prices;

3.1 Interference Constraints in Repacking and Feasibility Checking

The description of the FCC incentive auction DCA setting with the engineering constraints is available in detailed files on the FCC web site, which we used [11]. There are two csv data files which include

- A domain file called "Domain-2013July15.csv", of size 306KB, that specifies the feasible channels for each station.
- An interference file called "Interference-Paired-2013July15.csv", of size 6219KB, that specifies the interference constraints that the repacking must meet. This includes:
  - Pairs of (station, station) that must not be assigned to the same channel (among a given list of channels).
  - Pairs of (station, station) that must not be assigned to adjacent channels (among a given list of channels).

Let $\mathcal{S}$ be a set of stations that needs to be repacked into a list of channels in set $\mathcal{C}$. We use i, j as indices for stations and use k as indices for channels. Let $\mathcal{C}_i \subset \mathcal{C},\ i \in \mathcal{S}$, be the list of feasible channels to station i. Let $\mathcal{I}_c$ be the list of triplets (i, j, k) such that stations i and j cannot be assigned to the same channel k. Let $\mathcal{I}_a$ be the list of triplets (i, j, k) such that stations (i, j) cannot be assigned to channel $(k, k + 1)$ respectively. Data for $\mathcal{C}_i$, $\mathcal{I}_c$ and $\mathcal{I}_a$ are available from the domain file and the interference-paired file.

From a given list of channels $\mathcal{C}$, we say the set $\mathcal{S}$ of stations is feasible with respect to $\mathcal{C}$ if the stations can be packed into the channels without violating any of the constraints.

Let z_{ik} be a binary variable that indicates whether station i is assigned to channel k. We say $\boldsymbol{z}$ is an assignment to the repacking problem. For $\boldsymbol{z}$ to be feasible, we need the following: (a) all the indicator variables z_{ik} are binary, (b) each station is assigned to exactly one channel, and (c) no pairs of stations that might interfere with each other can be assigned to the same or adjacent channels. The feasibility function can be defined as

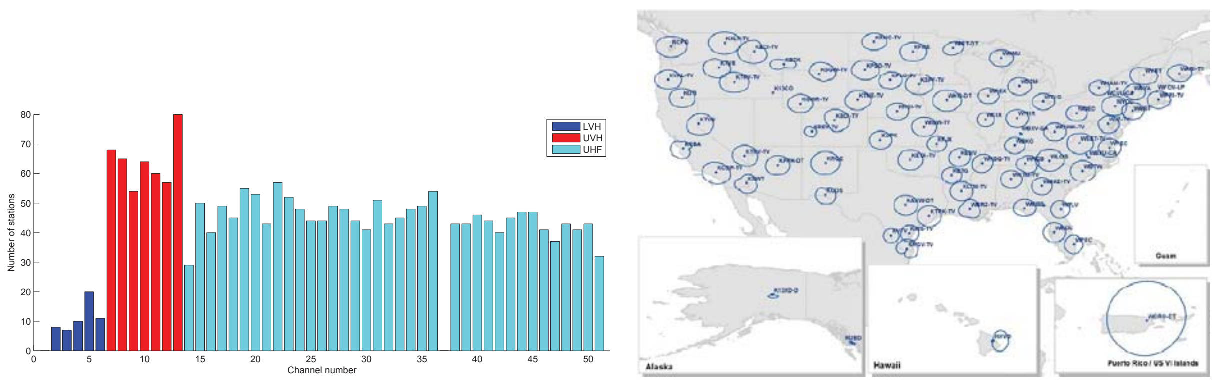
$$F(\mathcal{S}) = \begin{cases} 1, & \text{if } \mathcal{P}(\mathcal{S}, \mathcal{C}) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{P}(\mathcal{S}, \mathcal{C})$ is the set of feasible assignments of stations in the set $\mathcal{S}$ to a list of available channels $\mathcal{C}$ and is defined as

$$\mathcal{P}(\mathcal{S}, \mathcal{C}) \quad = \quad \left\{ \boldsymbol{z} : \quad \begin{array}{l} z_{ik} \in \{0,1\}, \forall i \in \mathcal{S} \text{ and } k \in C_i, \\ \sum_{k \in C_i} z_{ik} = 1, \forall i \in \mathcal{S}, \\ z_{ik} + z_{jk} \leq 1, \forall (i,j,k) \in \mathcal{I}_c(\mathcal{S}), \\ z_{ik} + z_{jk+1} \leq 1, \forall (i,j,k) \in \mathcal{I}_a(\mathcal{S}) \end{array} \right\}, \tag{13}$$

where $\mathcal{I}_c(\mathcal{S})$ and $\mathcal{I}_a(\mathcal{S})$ are corresponding subsets of $\mathcal{I}_c$ and $\mathcal{I}_a$ that involves only stations in $\mathcal{S}$.

To summarize the data and the feasibility problem, there are $n = 2177$ stations and $m = 49$ channels available (ranging from channel 2 to channel 51, with channel 37 not available). Figure 1-a shows the list of channels that range from 2 to 51 and the number of stations currently allocated to each channel. As described in [9], these channels are divided into two bands; very-high frequency (VHF) band between 54-216 MHz and ultra-high frequency (UHF) band between 614-698 MHz. These correspond to channels 2-13 in the VHF and channels 14-51 in the UHF. The VHF band is further divided into two bands; the Lower VHF, denoted as LVH, with channels 2-6 and the upper VHF, denoted as UVH, with channels 7-13. As can be seen in Figure 1-a, most of the stations are currently allocated in the UHF band (shown in cyan color). Figure 1-a shows the physical locations of all the stations that are currently allocated to channel 13.



(a) Current channel distribution

(b) Stations allocating to channel 13

Figure 1: Statistic of current channel allocation (source [10]).

The aim of reverse auctions is to clear a number of channels in the high-frequency band, say channels 33-51. This means all stations that are currently in this band need to either go off-air or be reallocated to lower-frequency channels 2-32. To avoid interference, stations who are currently in channels 2-32 also needs to either go off-air or be reallocated to different channels. The average number of feasible channels that each station can be allocated to is 44.15 (out of 49 channels) with most of the channels being freely allocated to any available channels. However, some stations only have a limited number of feasible channels (that is, there are stations with only three possible channel assignments).

The repacking problem can be viewed as a generalized graph-coloring problem where the nodes represent the stations, the edges represent the interfering constraints and the colors represent the channels. Let $G(N, E)$ denote the corresponding graph. This network has 2177 nodes and 3908 edges. Here, an edge is formed if there is any possible interference through any pairs of channels, i.e. could be the same or adjacent channels. Roughly speaking, the repacking problem is similar to a generalized graph-coloring problem where each node is assigned to a color among a predefined set of colors in such a way that no neighboring nodes have the same color. To model the interfering constraint in the repacking problem, the graph-coloring problem needs to be generalized to include constraints; one such constraint, for instance, is that some neighboring nodes cannot be colored with adjacent colors. Here, 'adjacency' is with respect to a predefined order of colors.

Figure 2-a shows the adjacency matrix of $G(N, E)$. Each dot at row i and column j in Figure 2-a appears if there is potential interference between stations i and j. Figure 2-b show the same interference matrix after the stations has been reorder in such a way that the non-zeros appear mostly on the diagonals.

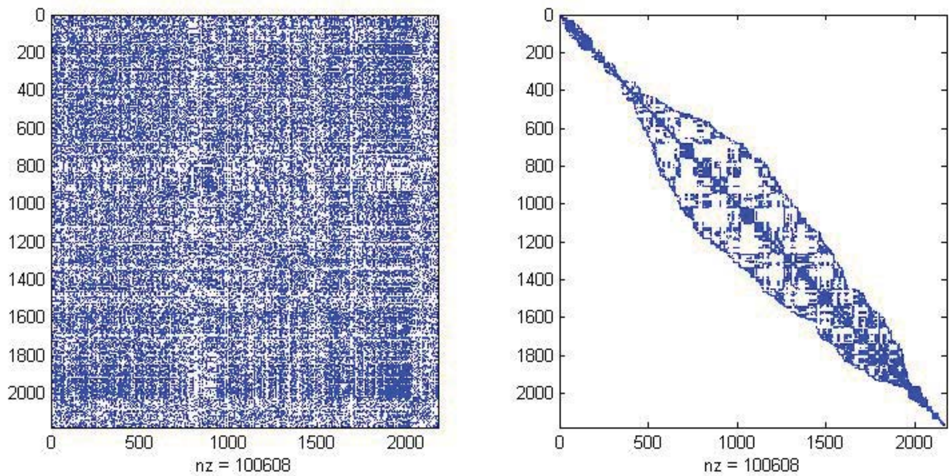


Figure 2: Interference matrix on the original ranking of stations from 1-2177 (on the left) and after reordering (on the right).

There are a large number—up to 2.9×10^6—of constraints requiring pairs of stations not to be allocated in the same or adjacent channels. Although this is smaller than $2mn^2 = 493 \times 10^6$ in the worst case, i.e., when interference matrices are fully dense, it is still a very large number. This makes checking the assignment feasibility very challenging for the full problem when all 2177 stations are rejected. In our experiments, however, the largest number of stations being rejected among all the instances tested is less than 1000 and hence CPLEX can still handle the feasibility problem. The feasibility problem does not involve an objective function and hence is much easier to solve than the winner determination problem in a VCG setting.

In the experiments, we used CPLEX to solve the repacking feasibility problem. We could also use a satisfiability (SAT) formulation for this purpose as has been done by Leyton-Brown [16]. Our choice of CPLEX here was for the convenience of implementation of the whole DCA process and due to some special network structural properties of the repacking problem that CPLEX could exploit. However, a discussion on the comparison between the performance of SAT and CPLEX is out of the scope of this manuscript since our focus is on the price setting and not on computational method for solving the feasibility problem.

3.2 Adapting the DCA to FCC Reverse Auctions Using OPT-SCHED

We adapt the general DCA framework of Algorithm 1 to incentive auctions. First, as in the homogeneous setting, we need to adapt Step 2.1 and set the target number of accepting bidders to $T^{(r)} = n^{(r)} - (n^{(r)} -$

$T)/(m-r)$, where $n^{(r)} = |\mathcal{A}^{(r)}|$, and solve the following model

$$\begin{aligned} \min_{\boldsymbol{y}} \quad & \sum_{i \in \mathcal{A}^{(r)}} y_i\big(y_i(u_i - l_i) + l_i\big), \\ \text{s.t.} \quad & \sum_{i \in \mathcal{A}^{(r)}} (1 - y_i) \le \frac{n_r - T}{m - r + 1}, \\ & 0 \le y_i \le 1, \forall i \in \mathcal{A}^{(r)}, \end{aligned} \tag{14}$$

to find a vector of prices to offer the bidders. Here, $y_i = (p_i - l_i)/(u_i - l_i)$ is a transformed variable that can be interpreted as the acceptance probability given the offer price p_i. The key difference between incentive auctions and the homogeneous setting is in the feasibility function. Specifically, the feasibility function in incentive auctions is a complicated function that is highly sensitive to the set of rejected stations. We need to adapt Step 2.2. of Algorithm 1 to find the set of new rejections and check the feasibility on the updated set of rejected stations. Details about this and other steps are presented as Algorithm 3.[4]

ALGORITHM 3: A DCA Framework using Optimal Price Setting for Incentive Auctions

Input: A set of stations $\mathcal{N} = \{1, \ldots, n\}$, an auctioneer with a feasibility function $F : 2^{\mathcal{N}} \to \{0, 1\}$. A target number of rounds allowed m. Initial valuation function estimates v_i.

Output: A set of feasible stations to reject $\mathcal{R} \subset \mathcal{N}$, i.e. $F(\mathcal{R}) = 1$, and the corresponding offer price vector $\boldsymbol{p}$ that aims to minimize the expected payment on the remaining stations.

1. Set the initial prices $\boldsymbol{p}$ at the reserves. Let the set of rejected bidders be $\mathcal{R}^{(r)} = \emptyset$;

for *round* $r = 1...m$ **do**

  2.1. Set the target number of accepting bidders $T^{(r)} = n^{(r)} - (n^{(r)} - T)/(m - r)$ where $n^{(r)} = |\mathcal{A}^{(r)}|$ and solve Model 14 to find a vector of prices $\boldsymbol{p}$ to offer the bidders;

  2.2. Find the set of rejected offers $\mathcal{R}$;

  if $F(\mathcal{R}^{(r)} \cup \mathcal{R}) = 1$, i.e. via solving 13, **then**

    2.2.1. $\mathcal{R}^{(r+1)} \leftarrow \mathcal{R}^{(r)} \cup \mathcal{R}$;

    2.2.2. Update the distributions of the bidders' valuations using Formulation 11;

  else

    2.2.3. Enter the final Step 3;

  end

end

3. Set all remaining bidders $\mathcal{N} \backslash \mathcal{R}^{(r)}$ as winners and pay them their offer prices;

In order to apply the models developed in Section 2 to finding the offer prices in each round, we need to modify the feasibility constraint. As the bidders' true values are random variables and unknown to the auctioneer, any fixed set of offer prices leads to a stochastic set of rejected bidders and hence the repacking feasibility is also stochastic.

However, at the beginning of each round, we could simulate the feasibility problem to draw a curve that shows the probability of having feasible repacking as a function of the number of new stations added. We could then choose a target $T^{(r)}$ so that the chance of feasibility when adding $T^{(r)}$ new stations is at some threshold (say 99% chance of feasibility). Once a target $T^{(r)}$ has been determined, we can then solve Model 5 to obtain the offer price. The choice of the feasibility probability (and hence $T^{(r)}$) would depend on how quickly or slowly the auctioneer wants to run the auction. For example, at the beginning of the auction, the auctioneer might want to have small $T^{(r)}$ for more accurate price discovery but then increase $T^{(r)}$ toward the end of the auction to lower the expected payment faster.

[4] In Step 3 of Algorithm 3, it is possible to do an improvement and select a smaller set of winners by solving a winner determination problem constrained by having $\mathcal{R}^{(r)}$ as a subset of rejected bids. That problem, however, has a similar structure to a winner determination problem arising from a single-round, sealed-bid auction and is difficult to solve due to the large set of engineering constraints.

Here we present a simple algorithm for price setting in incentive auctions. Within the scope of this paper, we do not undertake extensive simulation to obtain a cumulative function of the feasibility with respect to the number of new stations, $T^{(r)}$, added in each round. Instead, we use an estimated $T = 1177$ number of stations in the final set of active bidders, that is, to have $U = 1000$ stations feasibly rejected.[5] We compare the performance of our optimization model with the percentile-based method where both use the same U.

To deal with the heterogeneity of the stations with respect to the feasibility function, we propose to associate each active station with a weight that is proportional to the possibility of causing interference on other stations, especially those that have already been rejected. One possibility for setting such a weight vector is to set

$$w_i = d_i(\mathcal{N}) + d_i(\mathcal{R}^{(r)}), \forall i \in (\mathcal{N} \backslash \mathcal{R}^{(r)}), \tag{15}$$

where $d_i(\mathcal{N})$ is the number of stations in the entire set of stations $\mathcal{N}$ that i might interfere with, and $d_i(\mathcal{R}^{(r)})$ is the number of stations in the current rejected list that i might interfere on. By setting the weights higher for stations with a higher potential of interference on others, we essentially try to avoid having these stations rejected in the next rounds by offering them higher prices (recall Equation 8 and Proposition 3).

3.3 Advanced Model for Optimizing DCA Prices in FCC Reverse Auctions

The OPT-SCHED model shown in 14 proved to perform well when the bids are uniformly distributed among all bidders, as we show in the experimental results in Section 4.2.1 (Figure 5). We show in Section 4.2.2 that this is no longer the case if a large group of highly interfering stations, e.g. those from the same physical area, request large bids together. In that case, OPT-SCHED will attempt to reject these bids which could lead to infeasibility at the early rounds and hence the premature end of the auction.

To demonstrate this, consider the largest clique in the graph $G(N, E)$ described in Section 3.1 which is of size 46. In this case, we cannot reject all the stations in that large clique if we have less than 46 channels available. In general, consider a clique of stations $Q = \{S_1, S_2, \ldots, S_k\}$ where any pair of stations cannot be assigned to the same channel. From any such clique, there are, at most, $|\mathcal{C}|$ stations that can be assigned to $|\mathcal{C}|$ channels available. In each round, if we have already rejected Δ_r stations, then the maximum number of remaining stations that we can reject is $|\mathcal{C}| - \Delta_r$. We introduce the following constraint to Model 14:

$$\sum_{i \in Q} (1 - y_i) \leq \left(\frac{n_r - T}{m - r + 1} \right) \left(\frac{|Q|}{n_r} \right) \left(\frac{|\mathcal{C}| - \Delta_r}{|\mathcal{C}|} \right). \tag{16}$$

We call this a clique constraint and its interpretation is as follows. The first component of the R.H.S of Inequality 16 is very similar to that shown in the OPT-SCHED model. The second component is basically used to scale down the first term as the constraint is applied to stations in the clique of size $|Q|$ instead of to all the active bidders. The third component of the clique constraint is the interesting part. If none of of the stations $\{S_1, S_2, \ldots, S_n\}$ in clique Q has been rejected, then $\Delta_r = 0$ and the third component does not have any effect. However, as Δ_r gets larger through the rounds due to some of these stations being rejected, then the third component becomes smaller which effectively results in $y_i,\ i \in Q$ getting larger in the optimal solution. In other words, as there are more and more stations in the cliques get rejected, the offer prices to the remaining stations should be high enough to avoid further rejection because, otherwise, infeasibility is likely to occur.

The clique constraints essentially avoid the rejection of too many stations in the same large clique. For each clique, depending on how many stations in that clique have been rejected, the prices offered to

[5] This choice of U comes from the prior experiments we have on the feasibility problem.

Subnetwork	States	# Stations	# Cliques
1	WA, OR, ID, MT, WY, ND, SD, NE, MN, IA, GU, WI	363	437
2	CA, NV, AZ, UT, CO, NM, KS, OK, TX, AK, HI	602	566
3	ME, NH, VT, MA, RI, CT, NY	160	323
4	FL, GA, AL, MS, LA, PR	335	872
5	AR, SC, KY, TN, NC	238	610
6	OH, IL, IN, MI, MO	283	1008
7	PA, NJ, DE, MD, DC, WV, VA	196	447

Table 1: Network decomposition

the remaining stations would be adjusted accordingly. The following model, which we call OPT-SCHED-CLIQUES, shows the complete formulation after these clique constraints are added,

$$\begin{aligned} \min_{\boldsymbol{y}} \quad & \sum_{i \in \mathcal{A}^{(r)}} y_i\big(y_i(u_i - l_i) + l_i\big), \\ \text{s.t.} \quad & \sum_{i \in \mathcal{A}^{(r)}} (1 - y_i) \leq \frac{n_r - T}{m - r + 1}, \\ & \sum_{i \in Q} (1 - y_i) \leq \left(\frac{n_r - T}{m - r + 1}\right) \left(\frac{|Q|}{n_r}\right) \left(\frac{|\mathcal{C}| - \Delta_{r,Q}}{|\mathcal{C}|}\right), \forall Q \in \mathcal{Q} \\ & 0.5 \leq y_i \leq 1, \forall i \in \mathcal{A}^{(r)}, \end{aligned} \tag{17}$$

where $\mathcal{Q}$ is the set of all cliques, $\mathcal{C}$ is the number of channels available, and $\Delta_{r,Q}$ is the number of stations in clique Q that has been rejected by round r. In the last set of constraints in Model 17, we basically restrict the offer prices' reduction in each round to be no more than 50%. This makes the algorithm to behave more stably, especially in the initial rounds when the adjustment term $\left(\frac{|\mathcal{C}| - \Delta_r}{|\mathcal{C}|}\right)$ does not have much influence.

In the implementation of the algorithm, we use the Matlab code written by [27], which claims to implement the Bron-Kerbosch algorithm [3], to generate cliques. Since finding the set of all cliques for this entire graph of 2177 nodes would be too time-consuming, we decompose the network into seven subnetworks shown in Table 1. The numbers of stations and cliques in each subnetwork are shown in the third and the fourth columns, respectively. We then find the cliques within each subnetwork to form a total of 4263 cliques ranging in size from three to 46 (here we do not use cliques of size two).

Note that combining cliques from the subnetworks will exclude some cliques from the original big network (i.e. those cliques that join stations from more than one subnetworks). However, this exclusion is not a big issue since stations are usually located in large cities which means large cliques are usually formed by stations from the same areas. Decomposing the network into groups of states as shown in Table 1 results in not many large cliques being missed. Equally important, the division into subnetworks allows us to decompose the constraint set in Model 17 into seven subsets of separable constraints, one for the cliques in each subnetwork. From that, we can use a decomposition technique to solve Model 17 more efficiently.

It is also noted that the resulting model corresponds to a strictly convex quadratic program and hence the uniqueness property of the offer prices still reserves. In addition, we can also show the symmetry and monotonicity of the optimal solution in some senses. For example, any two stations that share symmetric characteristic on their value distributions and their interference constraints should be offered at the same prices. If everything is the same for two stations except for their upper bounds u, then the one with a higher upper bound should be offered at the higher price. Similar statement on the lower bounds can also be derived. A more interesting monotonicity property w.r.t the interference constraints can be stated as follows. For any two stations with the same value distribution while the set of clique constraints in one stations is 'contained' in the other, then the offer price to the station with more interference constraints should be higher than that of the other station. All these properties can be proven by exploiting the strict convexity of the objective function, the linearity of the constraints, and the positive sign of the coefficients that appear in the clique constraints. The proofs are, therefore, skip for the brevity of the paper.

3.4 Final Round Settlement in Incentive Auctions

Suppose that at round r, the set of rejected stations $R^{(r)}$ can still be feasibly repacked, i.e. $F(R^{(r)}) = 1$. The auctioneer then offers prices $p^{(r+1)}$ to the remaining active bidders $\mathcal{A}^{(r)}$. Let R_n be the set of newly rejected stations. The new set of all rejected stations is $R^{(r+1)} = R^{(r)} \cup R_n$. If these stations are repackable, i.e. if $F(R^{(r+1)}) = 1$, then we move to the next auction round. Otherwise, the auction goes to the final round settlement. There are several possible ways to settle here. One such way is to simply set stations in $\mathcal{A}^{(r)}$ as the winners and pay them the offer price $p^{(r)}$. A better way is to work with stations in the considering set R_n and find some intermediate offer prices within $[p^{(r+1)}, p^{(r)}]$. That way of settlement is, however, very problem-specific. For simplicity and consistency in comparison, we assume the settlement is designed in the following way. Among the set of all considering stations R_n, the auctioneer will choose a subset $X \subset R_n$ to reject such that $F(R^{(r)} \cup X) = 1$. Note that $F(R^{(r)}) = 1$ and hence $X = \emptyset$ is a feasible solution. However, we would also like to find X such that the payment to stations in $\mathcal{A}^{(r)} \backslash X$ is minimized; or equivalently, the payment to X is maximized.

For all stations $i \in (R^{(r)} \cup R_n)$, let z_{ik} be a binary variable that indicates whether station i is assigned to channel k. The following model solves for the optimal choice of X in the final round settlement,

$$\begin{aligned}
\min_{\boldsymbol{z}} \quad & \sum_{i \in R^{(r)} \cup R} p_i \left(1 - \sum_{k \in \mathcal{C}_i} z_{ik}\right), \\
\text{s.t.} \quad & \sum_{k \in C_i} z_{ik} = 1, \forall i \in R^{(r)}, \\
& \sum_{k \in C_i} z_{ik} \leq 1, \forall i \in R_n, \\
& z_{ik} + z_{jk} \leq 1, \forall (i,j,k) \in \mathcal{I}_c, \ \in R^{(r)} \cup R_n, \\
& z_{ik} + z_{jk+1} \leq 1, \forall (i,j,k) \in \mathcal{I}_a, \ \in R^{(r)} \cup R_n, \\
& z_{ik} \in \{0,1\}, \forall i \in R^{(r)} \cup R, \ k \in C_i.
\end{aligned} \tag{18}$$

Here, the first set of constraints ensures that each station in the existing list of rejected stations $R^{(r)}$ needs to be reallocated to a channel. The second set of constraints is similar except that the reallocation is optional to stations in R. The next two sets of constraints ensure that there is no interference among the repacked stations.

Problem 18 is very similar to the winner determination problem for reverse auctions in a VCG setting. The problem size here is, however, much smaller and hence CPLEX is able to handle it efficiently.

4 Experiments

In this section we instantiate the methodology in two settings, and present optimization experiments in both. We start with a setting where the items are homogeneous. We then proceed to the reverse auction in incentive auctions, using real FCC data.

We compare the performance of our method with a natural percentile-based method where the prices at round r are set to

$$p_i^{(r)} \quad = \quad l_i^{(r)} + \alpha_i (u_i^{(r)} - l_i^{(r)}), \tag{19}$$

where $\alpha_i = 1 - \frac{M}{n_r}$ and $M = \frac{n_r - T}{(m-r+1)}$ is the expected number of rejections per round. This essentially aims to distribute the expected number of rejections evenly among m rounds.

This percentile-based pricing scheme needs to be incorporated into a DCA such as in Step 2.1 of Algorithm 1. When we refer to a percentile-based method for the homogeneous setting, we mean Algorithm 2 with Step 2.1 being replaced by the pricing from Equation 19. Similarly, when we refer to a percentile-based method for incentive auctions, we mean Algorithm 3 with Step 2.1 replaced by the percentile-based pricing from Equation 19.

One can develop other percentile-based methods such as setting the offer prices at a fixed percentile in the bidders' value distributions, for example, either always aiming for a fixed 5% of rejection at each round or dynamically adjusting this percentile to be dependent on the current number of active bidders and the number of rounds remaining. We conducted extensive tests using these percentile-based methods. Their performance was almost the same on average as the percentile-based method described above. Therefore, we only present the performance of that percentile-based method in comparison with our optimization-based method.

4.1 DCA with Homogeneous Items

In the experiments, we let there be $n = 100$ bidders. We check the performance of the algorithms for various choices of the target T. We first generate random bounds for the bidders' valuations. The upper and lower bounds for bidder i are set to $u_i = (1+\delta)m_i$ and $l_i = (1-\delta)m_i$, where m_i is a uniform random variable in $[0,1]$. Here, δ is a measure of how good the auctioneer's estimate of the bidders' values is. We vary δ between 10% and 50%. We then draw $M = 10$ sample valuation vectors with bidder valuations from these ranges, that is, $\xi_i^{(k)} \sim U[l_i, u_i]$ for $k = 1, \ldots, M$, and for each bidder $i = 1, \ldots, n$. That gives us M auction instances to run on[6]. We report the average of them in the figures.

For convenience in reference, in the rest of the paper, we use the label 'OPT-SCHED' to refer to DCAs that make use of the price optimization method suggested by Model 9. These include output from Algorithm 2 for the homogeneous case and Algorithm 3 for incentive auctions (presented later in Section 3).

Figures 3-a and 3-b show the performance of the DCA for the setting where the number of rounds allowed $m = 50$, the weights $w_i = 1$ for all bidders i, the target $T = 50$, and $\delta = 20\%$.

The horizontal axis is the round number in the DCA. The vertical axis shows the number of active bidders, the total payment the auctioneer would pay if the auction ended at that round, the total value of the active bidders, and the optimal (lowest) payment, OPT, with which the auctioneer could procure the needed items if he knew the bidders' valuations.

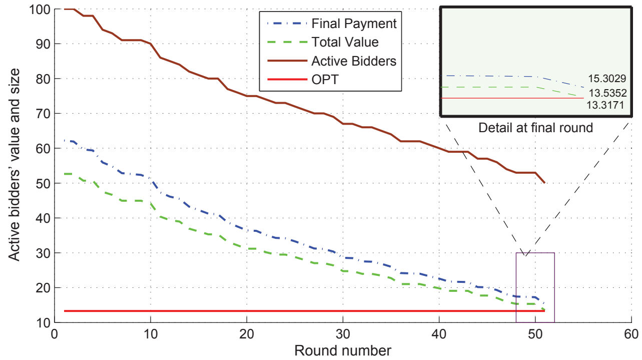
The number of active bidders, the total payment, and the total values decrease during the DCA as we expected. The total payment is always above the total value, and the total value is always higher than the optimal value. Comparing the final payments at the last round, one can see that our optimization-based approach significantly outperforms the percentile-based approach.

Figure 4 shows the final payoff as a function of the number of rounds allowed—for different targets T using the same parameters described earlier.

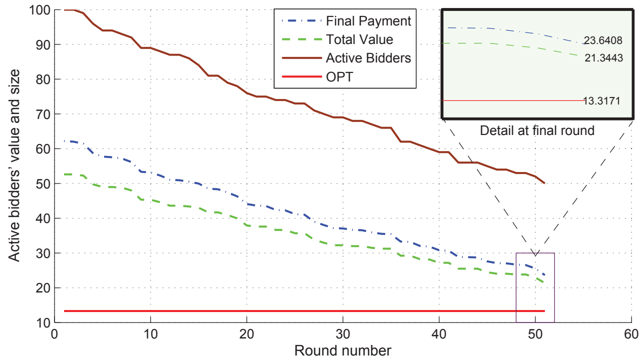
One can see that our optimization-based approach significantly outperforms the percentile-based approach across the board. Surprisingly, the final payment does not decrease when increasing the number of rounds allowed. This surprising finding does not always occur: we can find settings where the final payments for both the percentile-based method and the optimization-based method do keep decreasing as we allow more rounds. We can also design a specific percentile-based method that avoids this strange behavior completely, for example, by reducing offer prices of only the bidder with the highest upper bound first. However, that percentile-based method will not be effective in minimizing the final total payment.

We phrase the surprising behavior as a paradox and then proceed to explain it.

[6]Throughout the experiments, we use the first M fixed random seeds ranging from 1 to M for generating random samples. The usage of these fixed random seeds allows us to easier regenerate test instances in future work.

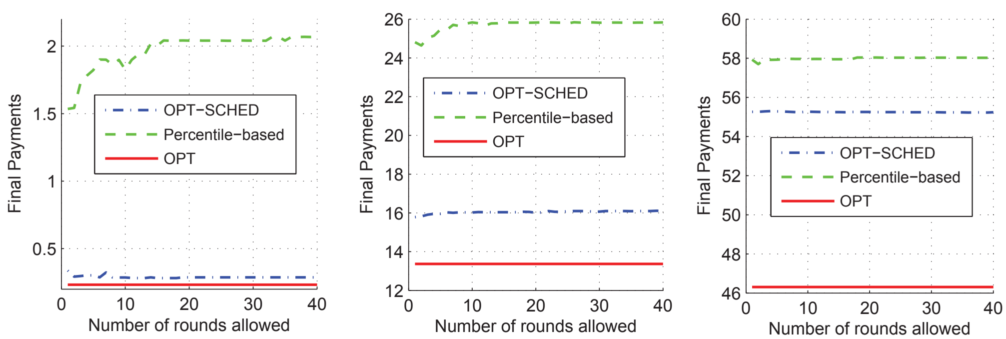


(a) OPT-SCHED algorithm



(b) Percentile-based algorithm

Figure 3: Comparison between OPT-SCHED and the percentile-based algorithm in terms of price discovery.



(a) T=5 (b) T=50 (c) T=95

Figure 4: Comparison between OPT-SCHED and the percentile algorithm in terms of the final DCA payments for a varying number of rounds allowed.

Paradox 1. *Having more refined offer prices can lead to a higher total payment.*

Explanation of the paradox: Throughout the multi-round auction, there are two essential effects on the final payment:

- Desirable effect (price discovery): The prices offered to the remaining active bidders keep decreasing; this lowers the final payment.
- Undesirable effect: Some lower-priced bidders are 'accidentally' rejected as their offer prices keep decreasing.

The first, desirable effect occurs in both the percentile-based method and the optimization-based method for price setting (OPT-SCHED). The second, undesirable effect occurs more in the percentile-based method and than in OPT-SCHED because the objective function in Model 10 has already aimed to lower this expected payment and hence high-priced bids are supposed to be rejected before others. This leads to overall better performance of OPT-SCHED compared to the percentile-based method, as shown in Figure 4. To summarize, the reason behind the paradox is that, due to the randomness of the bid values, having more rounds allow a higher chance of rejecting good bids before the final adjustment round. Once the good choices are excluded through more rounds, the less choice we have in solving the final settlement problem shown in Model 12, and thus the payment that the model is trying to minimize is higher.

We demonstrate this paradox through a specific example. Consider the following simple case with $n = 3$ bidders and a target number $T = 2$ items to be procured. Suppose all the bidders are still active at round r where the current offer prices are $p_i^{(r)}$. Consider the following two strategies:

- Strategy 1: Reduce all the prices in round $(r + 1)$ by the amounts δ_i for bidder i.
- Strategy 2: Reduce all the prices in round $(r + 1)$ by smaller amounts $\beta_i < \delta_i$ first, observe the bidders' responses, and then reduce the price by the additional amounts $(\delta_i - \beta_i)$ in round $(r + 2)$.

Consider the following scenario. Bidder 1 accepts new price $(p_1^{(r)} - \delta_1)$ but Bidders 2 and 3 do not accept $(p_2^{(r)} - \beta_2)$ and $(p_3^{(r)} - \delta_3)$. In this case, the number of acceptances is one for both strategies, so the auctions proceeds to the adjustment round.

- For Strategy 1, the adjustment round involves solving a knapsack problem to choose the two smallest offers among three choices $(p_1^{(r)} - \delta_1, p_2^{(r)}, p_3^{(r)})$.
- For Strategy 2, the adjustment round involves solving a knapsack problem to choose the two smallest offers among three choices $(p_1^{(r)} - \beta_1, p_2^{(r)}, p_3^{(r)})$.

4.2 DCA in Incentive Auctions

Since no incentive auctions have yet been conducted, we have to use simulated data on the bounds of the bidders' valuations. The bounds for the first experiment are generated using a uniform distribution where the upper and lower bound for bidder i are set to $u_i = (1 + \delta)m_i$ and $l_i = (1 - \delta)m_i$ and where m_i is a uniform random variable in $[0, 1]$. Here, $\delta = 0.2$ is a measure of how good the auctioneer's estimate of the bidders' valuations is. We then draw random sample bid values from these ranges, that is, $\xi_i \sim U[l_i, u_i]$ for each bidder $i = 1, \ldots, n$. We draw M = 10 valuation vectors. Each vector corresponds to a DCA instance. Each instance has one valuation per bidder. The setting for the second experiment is similar except that the mean value m_i is set proportional to the population that station i serves. Data for the populations served is obtained from [10].

4.2.1 Incentive Auctions with Homogeneous Bids

Figure 5 compares the average performance among OPT-SCHED, OPT-SCHED-CLIQUES, and Percentile-based for different choices on the number of rounds allowed. Figure 5-a shows the average final payoff to bidders versus the number of rounds allowed. One can see that both OPT-SCHED and OPT-SCHED-CLIQUES outperform the percentile-based approach on average for all choices of the number of rounds allowed. OPT-SCHED and OPT-SCHED-CLIQUES provide around 23% and 40% reduction, respectively, in the final payment compared to that of the percentile-based approach. These are results from the capability of the optimization-based approaches in rejecting high-value bids.

In Figure 5-b, the paradox of having the final payment increase as we increase the number of rounds allowed is again observable. It is noted though that, for all the three strategies, the sizes of the active bidders show decreasing trends as we increase the number of rounds allowed, as we expected. This is because the more rounds allowed, the higher refinement we have on the offer prices and hence the auctioneer has better knowledge of the truth bidders' values.

Figure 5-b also shows that the smallest average size of the active bidders in the percentile-based approach is 1650 while those for OPT-SCHED and OPT-SCHED-CLIQUES are 1550 and 1330, respectively. In other words, while the percentile-based approach is able to repack $(2177 - 1650 = 527)$ stations into 31 channels available, OPT-SCHED and OPT-SCHED-CLIQUES are able to repack 627 and 847 stations, respectively, into the same band of channels.

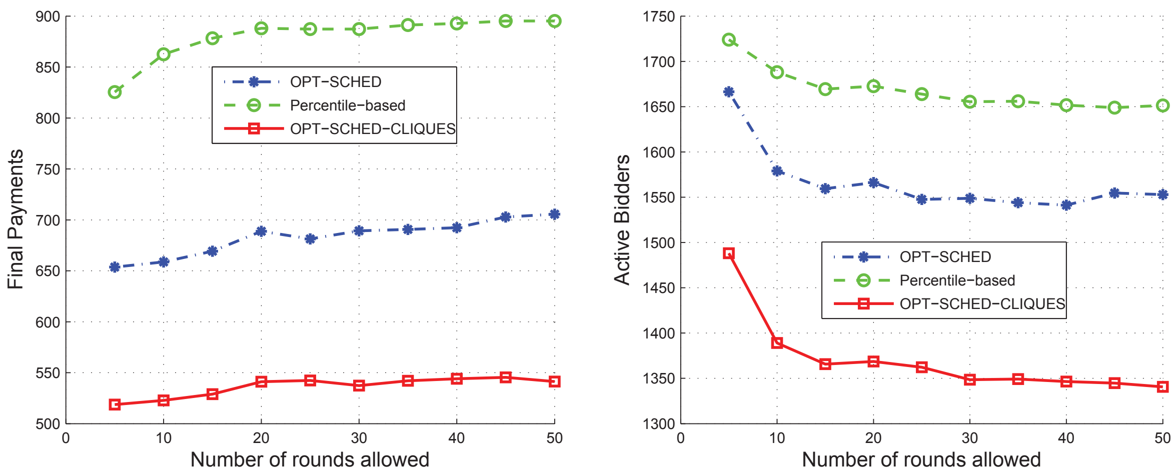


Figure 5: Final payments and active bidders using our optimization-based pricing strategies versus those of the percentile-based method—using real FCC engineering constraints data and using homogeneous bids.

Figures 6-a, 6-b and 6-c compare the performance of the DCA when using our optimization-based price-setting methods versus the percentile-based method, for the bids generated uniformly in $[0, 1]$, for the first sample DCA instance among M random instances generated. The number of rounds allowed is 50.

The horizontal axes of Figures 6-a, 6-b and 6-c show the round number in the DCA. The vertical axes in the three sub-figures show the number of active bidders, the total payment the auctioneer would pay if the auction ended at that round, and the total actual values of the active bidders, respectively. In each triplet of these curves, the dashed curves correspond to the percentile-based method, the dot-dashed curves to the OPT-SCHED, and the solid curves to the OPT-SCHED-CLIQUES. Because of the great overlaps between these curves, their ended points which correspond to the final rounds are marked with a circle (o), a star $(*)$ and a square box $(\square)$ respectively so that we can distinguish between them better. As can be seen, the number of active bidders, the payments and the total actual values decrease through the rounds. The payment is always above the total actual values, as we expected. Comparing the final payments at the last round, one can see that OPT-SCHED results in more stations to be reallocated and also in a

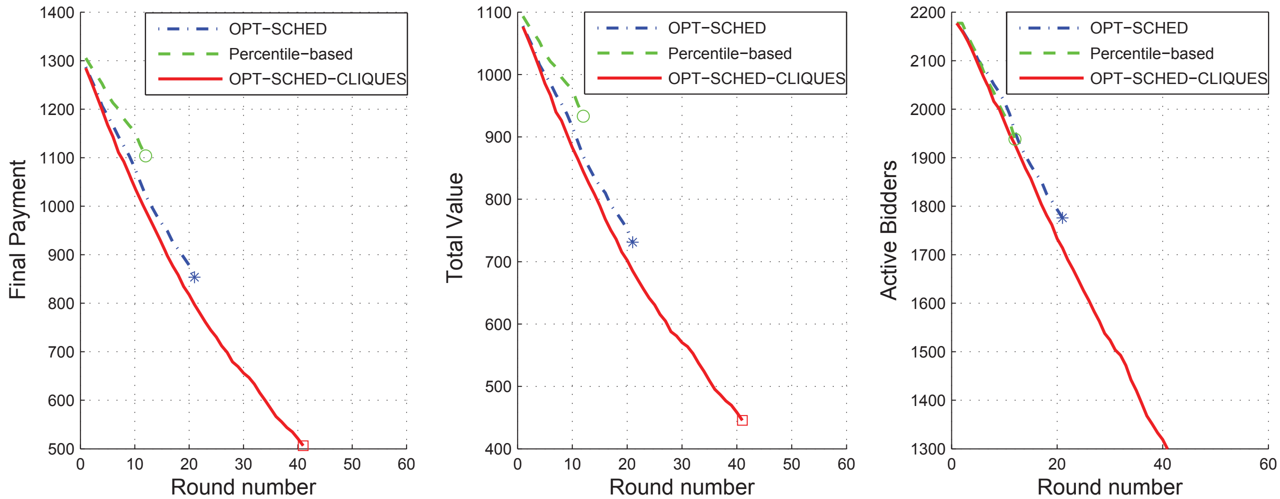


Figure 6: Comparison between optimization-based pricing strategies and those of the percentile-based method for one instance—using real FCC engineering constraints data and using homogeneous bids.

lower final payment compared to the percentile-based approach. The lower final payment of OPT-SCHED was partially due to it having more auction rounds before encountering infeasibility: the percentile-based method encountered infeasibility at round 12 while OPT-SCHED and OPT-SCHED-CLIQUES encountered it at rounds 21 and 43, respectively. The better performance of OPT-SCHED in dealing with infeasibility is due to the added weighted constraints which essentially avoid rejecting stations that are likely to cause interference. OPT-SCHED-CLIQUES further improves this by adding the clique constraints. Another reason why OPT-SCHED outperforms the percentile-based method is that it has a better way of rejecting high-priced bids. This effect can be seen in each fixed number of rounds. For example, if both algorithms are terminated at round 10, the payment for OPT-SCHED is 1070 while that for the percentile-based method is 1150.[7]

4.2.2 Incentive Auctions with Heterogeneous Bids

We now present a setting where OPT-SCHED has some issues when high-valued stations interfere with each other; for example, when groups of stations in the same area request high values. We demonstrate this through an experiment where the mean value m_i is set proportional to the population that station i serves. Figure 7 shows the average performance over $M = 10$ randomly generated auction instances.

In this case, OPT-SCHED yields higher payments on average than the percentile-based approach (although it yielded lower payments on the first two instances). In what follows, we provide some explanation for this. When setting the bids to be proportional to the population served, the bids from the stations in large cliques (e.g., stations in large cities) are often large since the populations served are large. If the number of rejected large bids is greater than the number of channels available, then we will reach infeasibility in repacking. Note that this issue is problematic for general DCAs where many small subsets of bidders can significantly affect the feasibility function. Among the random instances generated, the performance of the percentile-based approach was less affected by this because it uses a very conservative

[7] In the incentive auction setting we do not include data for the optimal payment if the auctioneer knew the bidders' valuations since the WDP is too difficult to solve to optimality according to Milgrom and Segal [19].

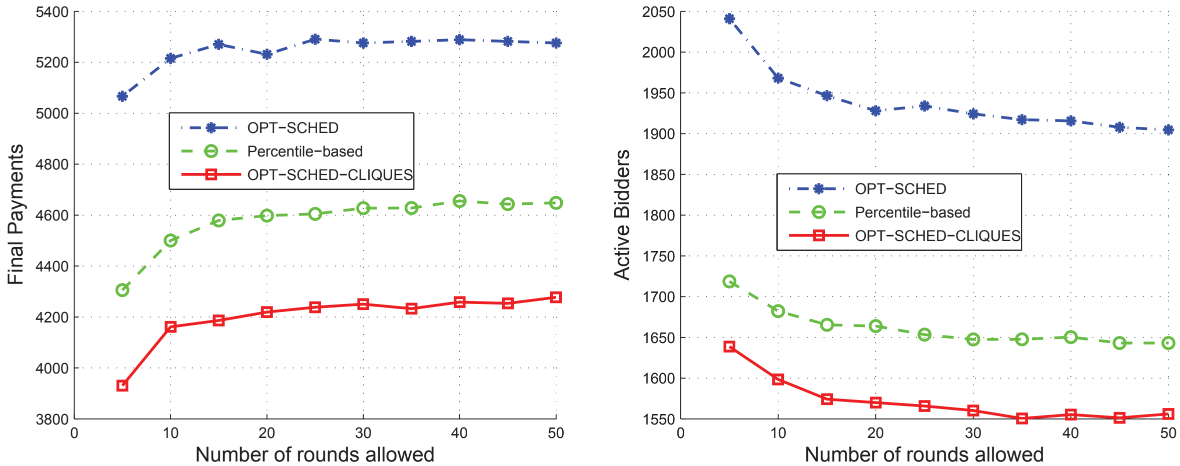


Figure 7: Final payments and active bidders using our optimization-based pricing strategies versus those of the percentile-based method—using real FCC engineering constraints data and using heterogeneous bids.

approach of gradually lowering prices for all bidders in the same way. By adding clique constraints on these subsets of bidders to ensure that no more than a certain number of stations in each large clique is rejected, the OPT-SCHED-CLIQUES resolves this issue and produces lower final payments and higher numbers of stations being repacked. Figures 8 and 9 show the same set of data compared to Figure 6 for two instances among M instances generated. OPT-SCHED performs better than the Percentile-based in the first instance shown in Figure 8 but the opposite is seen in the third instance shown in Figure 9. OPT-SCHED-CLIQUES outperforms both OPT-SCHED and the percentile-based approach in both cases.

We have also run experiments with other bidders' value distribution such as to set $m_i = Pop_i/deg_i$; i.e. the average bidders' values are proportional to the share of the population in the area. We find OPT-SCHED-CLIQUES outperforms the percentile-based approach. Figure 10 shows the performance comparison between the percentile-based approach and the OPT-SCHED-CLIQUES.

It is clear from Figure 10 that OPT-SCHED-CLIQUES outperforms Percentile-based in both the final payment and the number of stations feasibly repacked.

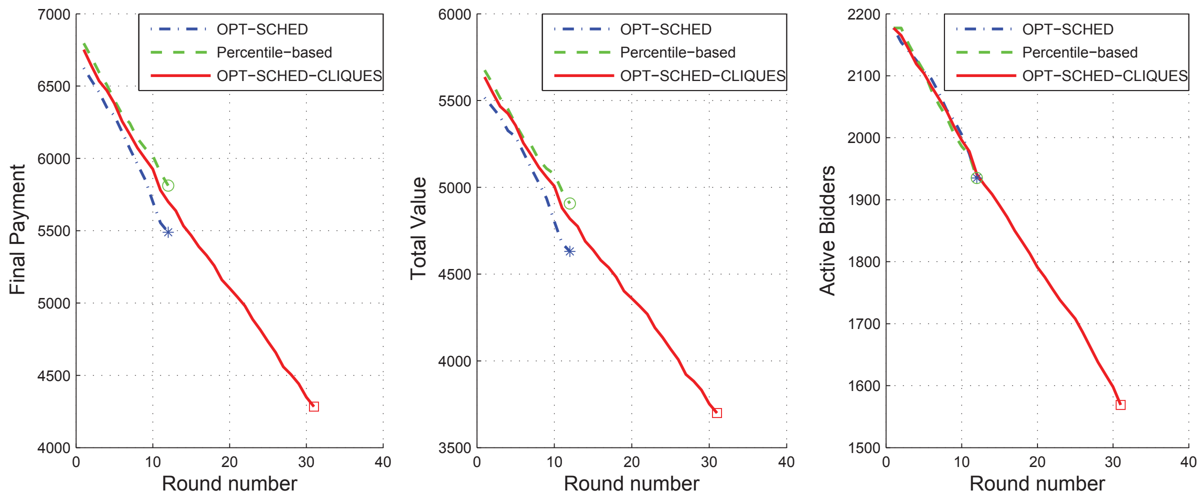


Figure 8: Comparison between optimization-based pricing strategies and those of the percentile-based method for instance number one—using real FCC engineering constraints data and using heterogeneous bids.

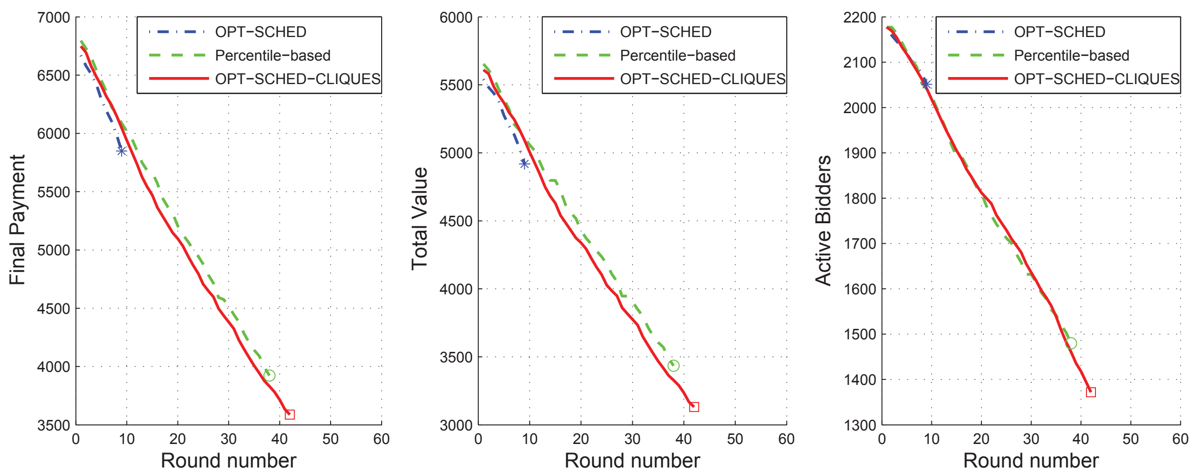


Figure 9: Comparison between optimization-based pricing strategies and those of the percentile-based method for instance number three—using real FCC engineering constraints data and using heterogeneous bids.

5 Conclusions and Discussion

A descending (multi-item) clock auction (DCA) is a mechanism for buying items from multiple potential sellers. DCAs have been proposed as the method for procuring spectrum from existing spectrum holders in the FCC's imminent incentive auctions so spectrum can be repurposed to higher-value uses. However,

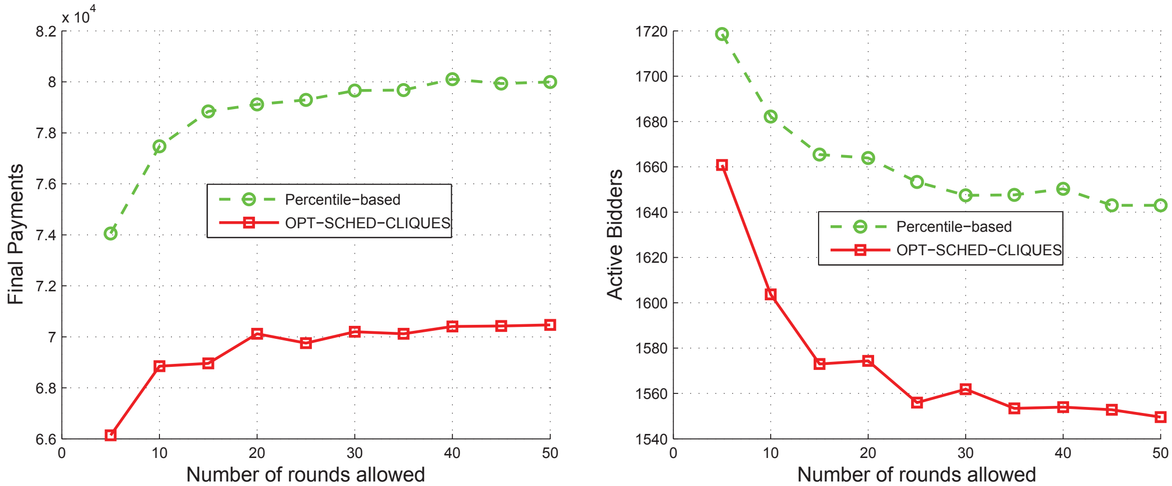


Figure 10: Final payments and active bidders using our optimization-based pricing strategies versus those of the percentile-based method—using real FCC engineering constraints data and using heterogeneous bids: POP/Size.

the DCA design has lacked a way to determine the prices to offer the bidders in each round. This is a recognized, important, and timely problem.

We presented, to our knowledge, the first techniques for this. We develop a percentile-based approach which provides a means to naturally reduce the offer prices to the bidders through the bidding rounds. We also develop a continuous optimization model for setting prices so as to minimize expected payment while stochastically satisfying the feasibility constraint. (The DCA has a final adjustment round that obtains feasibility after feasibility has been lost in the final round of the main DCA.) We proved attractive properties of this, such as symmetry and monotonicity. We developed certain kinds of clique constraint to add to the continuous optimization problem that capture feasibility better (and reduce the possibility of collusion among bidders). We developed computational methods for solving the model and very efficient polynomial-time algorithms. (We also developed optimization models with recourse, but they are not computationally practical.)

We presented experiments both on the homogeneous items case and the case of FCC incentive auctions, where we used real FCC interference constraint data to get a fully faithful model of feasibility. The experiments showed that our optimization-based price-setting approach significantly outperforms the natural percentile-based approaches in minimizing the final payment by the auctioneer. In incentive auctions, the optimization model helps feasibly repack more stations. An unexpected paradox on the performance of DCAs was that sometimes when the number of rounds allowed increases, the final payment can actually increase. We provided an explanation of this paradox.

There are a number of future research directions that can follow from this work. First, we find the paradox concerning the relationship between the expected payment and the number of rounds allowed quite intriguing. Further research on when this would occur, and on quantification of these relationships, would be interesting. In this paper, we have only considered the case where bidders' values are independent. It would be interesting to extend the techniques to settings with interdependent valuations. Further study to draw comparisons between this and the classical DCA framework is needed. Regarding the implementation of the reverse auctions DCA for the FCC incentive auctions, having extensive simulation for better estimation of the feasibility chance would improve the performance. Other ideas mentioned in the paper, such as using stochastic programming and chance constraints for the feasibility constraints, as well as approximate dynamic programming techniques, may also be worth pursuing for such a high-stakes setting. As has been done in all prior papers on incentive auctions, we studied the canonical problem where stations can either sell or keep their licenses. Recently is has become likely that the FCC will offer stations additional options as well during the reverse auction—in particular, moving to lower frequency bands. Those options may well have intermediate prices associated with them, and it is an important topic for future research how those prices, too, should be adjusted across rounds of the auction. We expect that optimization-based approaches may outperform simple heuristic approaches in that richer settings as well.

Bibliography

[1] Gediminas Adomavicius, Shawn P Curley, Alok Gupta, and Pallab Sanyal. 2012. Effect of information feedback on bidder behavior in continuous combinatorial auctions. *Management Science* 58, 4 (2012), 811–830.

[2] Patrick Bajari and Jungwon Yeo. 2009. Auction design and tacit collusion in FCC spectrum auctions. *Information Economics and Policy* 21, 2 (2009), 90–100.

[3] Coen Bron and Joep Kerbosch. 1973. Algorithm 457: finding all cliques of an undirected graph. *Commun. ACM* 16, 9 (1973), 575–577.

[4] Peter Cramton. 1997. The FCC spectrum auctions: an early assessment. *Journal of Economics and Management Strategy* 6, 3 (1997), 431–495. Special issue on market design and spectrum auctions.

[5] Peter Cramton and Jesse Schwartz. 2000. Collusive bidding: Lessons from the FCC spectrum auctions. *Journal of Regulatory Economics* 17 (2000), 229–252.

[6] Peter Cramton, Yoav Shoham, and Richard Steinberg. 2006. *Combinatorial Auctions*. MIT Press.

[7] Robert Day and Paul Milgrom. 2008. Core-selecting package auctions. *International Journal of Game Theory* 36, 3 (2008), 393–407.

[8] Robert Day and S Raghavan. 2007. Fair payments for efficient allocations in public sector combinatorial auctions. *Management Science* 53, 9 (2007), 1389–1406.

[9] FCC. 2012. *NOTICE OF PROPOSED RULEMAKING*. Technical Report. FCC. http://hraunfoss.fcc.gov/edocs_public/attachmatch/FCC-12-118A1.pdf.

[10] FCC. 2013a. *FCC baseline data and maps*. Technical Report. FCC. http://data.fcc.gov/download/incentive-auctions/OET-69/Baseline_Data_and_Maps_2013July.pdf.

[11] FCC. 2013b. FCC Repacking Constraint Files. (2013). http://data.fcc.gov/download/incentive-auctions/Constraint_Files/.

[12] FCC. 2014. The Path to a Successful Incentive Auction. (2014). FCC presentation 1/30/2014, http://wireless.fcc.gov/incentiveauctions/learn-program/Incentive_Auction_Jan_30_Present_9am.pdf.

[13] Marshall L Fisher. 2004. The Lagrangian relaxation method for solving integer programming problems. *Management Science* 50, 12 supplement (2004), 1861–1871.

[14] Thomas Hazlett, David Porter, and Vernon Smith. 2012. "Incentive Auctions" Economic and Strategic Issues. (2012). Arlington Economics white paper 6/12/2012.

[15] Frank Kelly and Richard Steinberg. 2000. A combinatorial auction with multiple winners for universal service. *Management Science* 46, 4 (2000), 586–596.

[16] Kevin Leyton-Brown. 2013. Investigating the Viability of Exact Feasibility Testing. (2013). Working Paper, University of British Columbia / Auctionomics, http://www.cs.ubc.ca/ ˜ kevinlb/talk.php?u=2013-Feasibility-Testing.pdf.

[17] Paul Milgrom. 2004. *Putting Auction Theory to Work*. Cambridge University Press.

[18] Paul Milgrom and Ilya Segal. 2012. Heuristic Auctions and U.S. Spectrum Repurposing. (2012). Presentation, Stanford University, www.stanford.edu/ ˜ isegal/fcc.ppt.

[19] Paul Milgrom and Ilya Segal. 2013. Deferred-Acceptance Heuristic Auctions. (2013). Working Paper, Stanford University, http://www.milgrom.net/downloads/heuristic.pdf.

[20] Marcelo Olivares, Gabriel Y Weintraub, Rafael Epstein, and Daniel Yung. 2012. Combinatorial auctions for procurement: An empirical study of the Chilean school meals auction. *Management Science* 58, 8 (2012), 1458–1481.

[21] Michael H Rothkopf, Aleksandar Pekeč, and Ronald M Harstad. 1998. Computationally manageable combinational auctions. *Management science* 44, 8 (1998), 1131–1147.

[22] Tuomas Sandholm. 2002. Algorithm for Optimal Winner Determination in Combinatorial Auctions. *Artificial Intelligence* 135 (Jan. 2002), 1–54.

[23] Tuomas Sandholm. 2006. Optimal Winner Determination Algorithms. In *Combinatorial Auctions*, Peter Cramton, Yoav Shoham, and Richard Steinberg (Eds.). MIT Press, 337–368. Chapter 14.

[24] Tuomas Sandholm. 2013. Very-Large-Scale Generalized Combinatorial Multi-Attribute Auctions: Lessons from Conducting $60 Billion of Sourcing. In *Handbook of Market Design*, Zvika Neeman, Alvin Roth, and Nir Vulkan (Eds.). Oxford University Press.

[25] Tuomas Sandholm, Subhash Suri, Andrew Gilpin, and David Levine. 2005. CABOB: A Fast Optimal Algorithm for Winner Determination in Combinatorial Auctions. *Management Science* 51, 3 (2005), 374–390.

[26] Tom Wheeler. 2013. The Path to a Successful Incentive Auction. (2013). FCC Chairman's post on the official FCC blog 12/6/2013.

[27] Jeffrey Wildman. 2013. Bron-Kerbosch maximal clique finding algorithm. (2013). http://www.mathworks.co.uk/matlabcentral/fileexchange/30413-bron-kerbosch-maximal-clique-finding-algorithm/content/maximalCliques.m.

APPENDIX

A.1 A Dynamic Programming Model for Optimal Price Setting in DCA

In each round of the descending clock auction, the auctioneer needs to offer each active bidder a price, i.e., to do Step 2.1. of Algorithm 1. Here we show a dynamic programming model that the optimal set of offer prices should solve.

Let $V(m, \mathcal{S}, \boldsymbol{u}, \boldsymbol{l})$ be the minimum expected payment that the auctioneer needs to pay to the bidders in a descending clock auction with m rounds, with a set of active bidders $\mathcal{S}$, with upper bounds $\boldsymbol{u}$ and lower bounds $\boldsymbol{l}$ within which the bidders' valuations lie. Let ξ be a realization of the bidders' values. For any offer prices $\boldsymbol{p}$ in the first round, the state of the auction by the end of that first round will be as follows.

- The number of rounds left will be $(m-1)$.
- The remaining active bidders will be $\mathcal{S}(\boldsymbol{p}, \xi)$. This includes bidder i if the offer price p_i is no smaller than the bidder's value ξ_i, i.e., $p_i \geq \xi_i$.
- A new vector of upper bounds $\boldsymbol{u}(\boldsymbol{p}, \xi)$ which updates the upper bound of any remaining active bidder i to x_i.
- Unchanged lower bounds $\boldsymbol{l}$.

The minimum expected value that the auctioneer needs to pay under the new state of the auction will be $V(m-1, \mathcal{S}(\boldsymbol{p}, \xi), \boldsymbol{u}(\boldsymbol{p}, \xi), \boldsymbol{l})$. Thus, the auctioneer's problem in the first round is to choose $\boldsymbol{p}$ that minimizes the expectation of $V(m-1, \mathcal{S}(\boldsymbol{p}, \xi), \boldsymbol{u}(\boldsymbol{p}, \xi), \boldsymbol{l})$. We have the Bellman optimality equation

$$V(m, \mathcal{S}, \boldsymbol{u}, \boldsymbol{l}) = \min_{\boldsymbol{p}} \qquad E[V(m-1, \mathcal{S}(\boldsymbol{p}, \xi), \boldsymbol{u}(\boldsymbol{p}, \xi), \boldsymbol{l})].$$

Solving this dynamic program would be extremely difficult. In fact, just finding $V(m, \mathcal{S}, \boldsymbol{u}, \boldsymbol{l})$ for the case $m = 1$ would be very difficult as shown in a simple case below.

A.2 Optimal Price Setting in the Last Round with Recourse Action

Consider the problem of setting prices in the final round of a descending clock auction. Assume that the actual bid values are uniformly distributed random variables, i.e., $\xi_i \sim U[l_i, u_i]$, where $(l_i, u_i), i = 1, \ldots, n$ are known. Let $\boldsymbol{p}$ be a vector of prices that the auctioneer offers to the bidders. Given the offer prices, the bidders might accept or reject the offers. The auctioneer then updates the best upper bounds on the bids values, that is, upper bounds for accepting bidders will be updated to the offer prices while those of rejected bidders will remain unchanged. The auctioneer chooses T bids with the smallest updated upper bounds and pays each of these bidders those prices. Since the bidders' values are random variables, the acceptance of the bidders for each set of offer prices $\boldsymbol{p}$ will also be stochastic, so the final payment is stochastic. We consider the problem of finding the optimal offer prices $\boldsymbol{p}$ such that the expected final payment is minimized. Here expectation is taken over the randomness of the bidders' valuations.

For convenience in notation, we perform a linear transformation on the price vectors $\boldsymbol{p}$ to $\boldsymbol{x}$ where $x_i = \frac{p_i - l_i}{u_i - l_i}$, that is, $x_i \in [0, 1]$ can be interpreted as the target chance of acceptance for bidder i. We also have $p_i = l_i + x_i(u_i - l_i)$. Let us denote by $f(\boldsymbol{x})$ the stochastic payment.

Let us first consider the simple case where $T = 1$ and $n = 2$. Here the payment is $min(u_1, u_2)$ if both bidders reject the offers, $min(p_1, p_2)$ if both of them accept, and $p_i, i = \{1, 2\}$ if only bidder i accepts the offer. The probability for each of these four events can be calculated as functions of $\boldsymbol{x}$. For example, the chance of rejecting both offers is $(1-x_1)(1-x_2)$. Putting all of these together, we have

$$f(\boldsymbol{x}) = \begin{cases} min(u_1, u_2), & \text{w.p. } (1-x_1)(1-x_2), \\ min(l_1 + x_1(u_1 - l_1), l_2 + x_2(u_2 - l_2)), & \text{w.p. } x_1 x_2, \\ l_1 + x_1(u_1 - l_1), & \text{w.p. } x_1(1-x_2), \\ l_2 + x_2(u_2 - l_2), & \text{w.p. } (1-x_1)x_2. \end{cases}$$

The expected payment is

$$\begin{aligned} E[f(\boldsymbol{x})] \quad = \quad & (1-x_1)(1-x_2)min(u_1, u_2) + x_1 x_2 min(l_1 + x_1(u_1 - l_1), l_2 + x_2(u_2 - l_2)) + \\ & x_1(1-x_2)(l_1 + x_1(u_1 - l_1)) + (1-x_1)x_2(l_2 + x_2(u_2 - l_2)). \end{aligned}$$

The problem of determining the optimal offer prices can therefore be formulated as

$$\begin{aligned} \min_{x_1, x_2} \quad & E[f(\boldsymbol{x})] \\ \text{s.t.} \quad & 0 \le x_i \le 1, \forall i = 1, 2, \end{aligned}$$

which is a non-convex quadratic optimization problem. If we extend the problem to the case $n > 2$, the problem becomes a polynomial optimization problem as follows;

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \sum_{\mathcal{S} \subset \mathcal{N}, \mathcal{S} \neq \emptyset} \left[\prod_{i \in \mathcal{S}} x_i \prod_{i \notin \mathcal{S}} (1 - x_i) \min_{i \in \mathcal{S}} \{ l_i + x_i(u_i - l_i) \} \right] + \prod_{i \in \mathcal{N}} (1 - x_i) \min_{i \in \mathcal{N}} u_i \\ \text{s.t.} \quad & 0 \le x_i \le 1, \forall i = 1, \dots, n, \end{aligned}$$

which is very difficult to solve. Notice that we have considered only the simple case of $T = 1$ and also considered finding the optimal decision in the last round only.

A.3 Proof of Proposition 5

Proof. For each $m \in \{0, 1, \dots, n\}$ and for each budget $B \ge 0$, let us define

$$\begin{aligned} V(m, B) = \min_{\boldsymbol{p}} \quad & \sum_{i=1}^{m} F_i(p_i) p_i, \\ \text{s.t.} \quad & \sum_{i=1}^{m} \omega_i F_i(p_i) \ge B, \\ & p_i \in \{P_{i1}, \dots, P_{i,k}\}, \forall i = 1, \dots, m. \end{aligned}$$

Then we have

$$\begin{aligned} V(m, B) = \min_{p_m} \quad & F_m(p_m) p_m + V(m-1, B - F_m(p_m)), \\ \text{s.t.} \quad & p_m \in \{P_{m1}, \dots, P_{m,k}\}, \end{aligned} \tag{20}$$

where $V(0, B) = 0, \forall B$. Suppose $F_j(p_j)$ receives one of $(L+1)$ values in the set $\{0, 1/L, \dots, 1\}$. Then we can calculate $V(1, B)$ for all $B \in \{0, 1/L, \dots, 1\}$. If we knew $V(m-1, B), \forall B \in \{0, 1/L, \dots, m-1\}$, then we can plug this in into Formulation 20 and obtain $V(m, B)$ by taking K calculations for $F_m(p_m)p_m + V(m-1, B - F_m(p_m))$ for each $p_m \in \{P_{m1}, \dots, P_{m,k}\}$ and then choose the minimum, i.e., $2K$ operations in total. To obtain $V(m, B)$ for all possible $B \in \{0, 1/L, \dots, m\}$, we would need to repeat this Lm times, which means the total operations incurred for each m is $2KLm$. Summing this for all $m \in \{1, \dots, n\}$ would require $KLn(n-1)$ operations. Thus the complexity of the algorithm is $\mathrm{O}(KLn^2)$. □

A.4 Approximation Method for the Multi-Round Case

First, assuming that the auctioneer has only one round left. Then the optimal prices to offer to the bidders will be the solution of Model 9 for the continuous case (or 10 for the discrete case). Now, given that the auctioneer has multiple rounds to do price discovery, he would not offer these 'optimal prices' right away. Instead, a set of higher prices will be offered first to learn more about the bidders' valuations and to update the bounds.

A simple way that the auctioneer can do this is to discretize the prices into m equal intervals between the uppers bounds and $\boldsymbol{p}^*$ and offer these to the bidders sequentially until feasibility does not hold.

A better way is to do this dynamically as shown in Algorithm 4. Here, after solving Model 9 (or 10) in Step 1, the auctioneer can offer a guess $p_i = \frac{u_i+(m-1)p_i^*}{m}$ to bidder i and see how the bidder responds. This price is obtained under the expectation that the offer price in the next m rounds will be distributed evenly within the range $[p_i^*, u_i]$. Notice, however, that once the auctioneer has offered the prices to the bidders and received their responses to form the new state of the auction, the auctioneer now has better information and can repeat Step 2.1 of Algorithm 1 to find the new set of offer prices, that is, to run Algorithm 4 again with the updated information. Formally:

ALGORITHM 4: Finding Offer Prices in Round m

Input: Current round r, a current set of active bidders $\mathcal{A}^{(r)}$, most up-to-date valuation estimates v_i.
Output: An offer price vector $\boldsymbol{p}$.
1. Solve Model 9 to obtain the optimal offer prices $\boldsymbol{p}^*$ as if this were the last round;
2. Divide the range $[p_i^*, u_i]$ into m equal intervals and set the actual offer prices $p_i = \frac{u_i+(m-1)p_i^*}{m}$;
